

# Construction of schemes over $\mathbb{F}_1$ , and over idempotent semirings: towards tropical geometry

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## Abstract

In this paper, we give some categorical description of the general spectrum functor, defining it as an adjoint of a global section functor. The general spectrum functor includes that of  $\mathbb{F}_1$  and of semirings.

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## 0 Introduction

The goal of this paper is to provide an explicit description of the mechanism of the spectrum functor; how we construct a topological space and a structure sheaf from an algebraic object, and why it behaves so nicely.

At first, we were only planning to give some definitions of schemes constructed from idempotent semirings motivated from the tropical geometry (cf. [IME]), since still there is no concrete definition of higher dimensional tropical varieties: in [Mik], Mikhalkin is giving a definition, but is very complicated and not used in full strength yet, so we cannot say that the definition is appropriate yet. Also, the algebraic theories of semirings are still inadequate, compared to those of normal algebras, though there are already many demands from various workfields. (cf. [Lit]). Even the tensor products have been constructed only recently (cf. [LMS]). So, we had to start from the beginning: giving a foundation of idempotent algebras.

Later, we noticed that there should be a comprehensive theory including the theory of schemes over  $\mathbb{F}_1$ , of which field is now in fashion (cf. [TV], [CC], [PL], etc.).

We chose the definition of commutative idempotent monoid to be infinitely additive. There are two reasons. First, if we restrict our attention to finite additivity, then the existence of all adjoint functors are already assured, by the general theory of algebraic systems. Secondly, we can handle topological spaces in this framework: a sober space can be regarded as a semiring with idempotent multiplication. Also, we can make it clear how we obtain an infinite operation (i.e. the intersection of closed subsets) from algebras with only finitary operators.

This fact shows that, there are adjoints between the three categories: that of algebraic complete idealic semirings, that of algebraic idealic schemes, and that of algebraic sober spaces:

$$(\mathbf{alg.IRng}^\dagger) \rightleftarrows (\mathbf{alg.ISch})^{\mathrm{op}} \rightleftarrows (\mathbf{alg.Sob})^{\mathrm{op}}.$$

This indicates that, the genuine theme of algebraic geometry and arithmetic geometry (including schemes over  $\mathbb{F}_1$ ) is no longer the investigation of the link between the category of algebras and the category of topological spaces, but that of the link between the category of algebras and the category of idempotent semirings. Roughly speaking, the whole workfield may be thought as contained in the tropical geometry, if you take the tropical geometry as a study of semirings.

After a while, we realized that there are strong resemblance between the theory of lattices and the argument in the first half of this article. We must admit that there are no new ideas contained in this article; most of the techniques and theorems have been already established decades ago. Therefore, readers who attempt to study in this workfield is strongly recommended to overlook the lattice theory (for example, [MMT], [JH]). However, we decided to contain definitions and propositions as self-contained as possible, since there seems to be some discrepancy of languages between each workfields, and one of the purpose

of this paper is to introduce connections between the lattice theory, tropical geometry, and schemes over  $\mathbb{F}_1$ .

We were irresolute for a while, of which definition of schemes would be the most general and natural one. For this, we decided to give the weakest (in other words, the most general) definition in which the global section functor  $\Gamma$  becomes an adjoint. This clarifies the condition of which algebra could give a topological space endowed with a structure sheaf.

Also, we know that we can further extend the arguments and constructions in this article by replacing commutative monoids with symmetric monoidal categories, but we dared not, for it would be too abstract for the present demands.

§1 is a summary of algebras and complete algebras. The definition and arguments are all standard – in other words, preliminary – in the lattice theory. In §2, we deal with  $R$ -modules, where  $R$  is a complete semiring. The results cannot be deduced from §1, so we had to argue separately. In §3, we focus on congruence relations, which is important when considering localizations. In §4 we deal with topological spaces, defining the spectrum functor by an adjoint. This is also standard in the lattice theory. §5 may be the most valuable part, in which we define  $\mathcal{A}$ -schemes, and define the spectrum functor (not the same one of §4, but also endows the structure sheaf to the underlying space) as an adjoint. All the adjoints we have constructed are illustrated in the end of 5.3.

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## 0.1 Notation and conventions

Throughout this paper, all monoids are assumed associative, unital, commutative.

We fix a universe, and all sets and algebraic objects are elements of this universe. For any set  $X$ , we denote the power set of  $X$  by  $\mathcal{P}(X)$ .

We frequently make use of the notation  $\sum^{<\infty}$ : this means that given a infinite sum, there exists a finite subindex which preserve the equality and inequality. For example,

$$x \leq \sum_{\lambda} a_{\lambda} \Rightarrow x \leq \sum_{\lambda}^{<\infty} a_{\lambda}$$

means that, if  $x < \sum_{\lambda} a_{\lambda}$ , then there are finitely many  $\lambda_1, \dots, \lambda_n$  such that  $x < \sum_{i=1}^n a_{\lambda_i}$ .

We also make use of the notation  $\cup^{<\infty}$ .

## 0.2 Preliminaries

- (1) We frequently make use of the categorical languages in [CWM], especially that of adjoint functors: Let  $U : \mathcal{A} \rightarrow \mathcal{C}$ ,  $F : \mathcal{C} \rightarrow \mathcal{A}$  be two functors

between two categories  $\mathcal{A}$  and  $\mathcal{C}$ . We say  $F$  and  $U$  are *adjoint*, or  $F$  is the *left adjoint* of  $U$ , if there is a natural isomorphism of functors

$$\mathrm{Hom}_{\mathcal{C}}(c, Ua) \simeq \mathrm{Hom}_{\mathcal{A}}(Fc, a) : \mathcal{C}^{\mathrm{op}} \times \mathcal{A} \rightarrow (\mathbf{Set}).$$

This is equivalent to saying that there are natural transformations  $\epsilon : \mathrm{Id}_{\mathcal{C}} \Rightarrow UF$  (the *unit*), and  $\eta : FU \Rightarrow \mathrm{Id}_{\mathcal{A}}$  (the *counit*) which satisfy  $\eta F \circ F\epsilon = \mathrm{Id}_F$  and  $U\eta \circ \epsilon U = \mathrm{Id}_U$ :

$$\begin{array}{ccc} F & & U \\ F\epsilon \downarrow & \searrow \epsilon U & \downarrow U\eta \\ FUF & \xrightarrow{\eta F} & F \\ & & U \end{array}$$

- (2) We will briefly overview the general theory of algebraic varieties (in the lattice theoretic sense). Here we also use the language of [CWM]. An *algebraic type*  $\sigma$  is a pair  $\langle \Omega, E \rangle$ , where  $\Omega$  is a set of finitary operators and  $E$  is a set of identities. Let  $D$  be the set of *derived operators*, i.e. the minimal set of operators including  $\Omega$ , and closed under compositions and substitutions. A  $\sigma$ -algebra is a set  $A$ , with an action of  $\Omega$  on  $A$ , satisfying the identities. A *homomorphism of  $\sigma$ -algebras* is a map  $f : A \rightarrow B$  between two  $\sigma$ -algebras, preserving the actions of  $\Omega$ . We denote by  $(\sigma\text{-}\mathbf{alg})$  the category of small  $\sigma$ -algebras. Here are some facts about algebraic types:

- (a) The category  $(\sigma\text{-}\mathbf{alg})$  is small complete and small co-complete.
- (b) Let  $f : \sigma \rightarrow \tau$  be a *homomorphism of algebraic types*, i.e.  $f$  is a map  $\Omega_{\sigma} \rightarrow D_{\tau}$  of operators which satisfies  $f(E_{\sigma}) \subset E_{\tau}$ . We say that  $\tau$  is stronger than  $\sigma$  in this case, and denote by  $\sigma \leq \tau$  for brevity. Then, the underlying functor  $U : (\tau\text{-}\mathbf{alg}) \rightarrow (\sigma\text{-}\mathbf{alg})$  has a left adjoint. In particular, there is a functor  $F : (\mathbf{Set}) \rightarrow (\sigma\text{-}\mathbf{alg})$  for any algebraic type  $\sigma$ , and the  $\sigma$ -algebra  $F(S)$  is called the *free  $\sigma$ -algebra* generated by  $S$ .

## 1 Complete algebras

Almost all the results are already established in the lattice theory. Here, we are just translating it into a somewhat algebro-geometric language. The reader who is well familiar with the lattice theory may skip.

### 1.1 Complete sets

**Definition 1.1.1.** (1) A *multi-subset* of a set  $X$  is simply, a map  $f : \Lambda \rightarrow X$  from any set  $\Lambda$ . We denote by  $\overline{\mathcal{P}}(X)$  the set of multi-subset of  $X$  modulo isomorphism. Often, a multi-subset is identified with its image.

- (2) A map  $\sup : \overline{\mathcal{P}}(X) \rightarrow X$  is a *supremum map* if it satisfies:

- (a) The map factors through the power set of  $X$ , namely, there is a map  $\mathcal{P}(X) \rightarrow X$  making the following diagram commutative:

$$\begin{array}{ccc} \overline{\mathcal{P}}(X) & \xrightarrow{\sup} & X \\ \text{Im} \downarrow & \nearrow & \\ \mathcal{P}(X) & & \end{array}$$

where  $\text{Im}$  sends a map  $\Lambda \rightarrow X$  to its image.

- (b) It is (infinitely) idempotent: It sends a constant map (from a non-empty set) to the unique element in its image.
- (c) It is (infinitely) associative: If  $f : \Lambda \rightarrow X$  and  $f_i : S_i \rightarrow X$  are multi-subsets of  $X$  satisfying  $\text{Im} f = \cup_i \text{Im} f_i$ , then  $\sup f = \sup\{\sup f_i\}_i$ .

We often write  $x \oplus y$  instead of  $\sup\{x, y\}$ . Note that when given a (not necessarily infinite) associative commutative idempotent operator  $\oplus$  on  $X$ , we can define a preorder on  $X$ , by defining

$$a \leq b \Leftrightarrow a \oplus b = b.$$

Conversely, the supremum map gives the supremum with respect to this preorder. The maximal element  $1$  of  $X$  is the *absorbing element* of  $X$ , i.e.  $\sup(S \cup \{1\}) = 1$  for any subset  $S$  of  $X$ . Also, there is an infimum of any subset  $S$  of  $X$ :

$$\inf S = \sup\{x \in X \mid x \leq s \text{ for any } s \in S\},$$

and the minimal element  $0$  of  $X$  is the *unit*, i.e.  $\sup S \cup \{0\} = \sup S$  for any subset  $S$  of  $X$ .

- (3) A *complete idempotent monoid* is a set endowed with a supremum map.
- (4) Let  $A, B$  be two complete idempotent monoids. A map  $f : A \rightarrow B$  is a *homomorphism of complete idempotent monoids* if it commutes with the supremum map:

$$\begin{array}{ccc} \overline{\mathcal{P}}(A) & \xrightarrow{f_*} & \overline{\mathcal{P}}(B) \\ \sup \downarrow & & \downarrow \sup \\ A & \xrightarrow{f} & B \end{array}$$

and sending  $1$  to  $1$ . Note that when this holds,  $f$  sends  $0$  to  $0$ , since  $0 = \sup \emptyset$ . We denote by **(CIM)** the category of complete idempotent monoids.

The notion "complete idempotent monoid" is used in [LMS], but in the lattice theory, this is referred to as a *complete lattice*. We will use the former notation.

**Definition 1.1.2.** The algebraic type  $\sigma$  with a binomial operator  $+$  is *pre-complete* if:

- (1)  $+$  is an associative, commutative idempotent operator with a unit 0 and absorbing element 1,
- (2) There are infimums for any two elements  $a, b$ :  $\inf(a, b) \leq a, b$ , and if  $x \leq a, b$  then  $x \leq \inf(a, b)$ .
- (3) If  $\phi$  is another  $n$ -ary operator, then it is  $n$ -linear with respect to  $+$ :

$$\begin{aligned} \phi(x_1, \dots, x_i + x'_i, \dots, x_n) \\ = \phi(x_1, \dots, x_i, \dots, x_n) + \phi(x_1, \dots, x'_i, \dots, x_n). \end{aligned}$$

The condition (1) and (2) are equivalent to saying that a  $\sigma$ -algebra is a *distributive lattice* with respect to  $+$  and  $\inf$ .

Note that any complete idempotent monoid is already pre-complete, if we restrict the supremum map to finite subsets. Also, see that  $\inf$  is defined algebraically, i.e.

- (a)  $a + \inf(a, b) = a$ .
- (b)  $\inf(x + a, x + b) + x = \inf(x + a, x + b)$ .

**Definition 1.1.3.** Let  $\sigma$  be a pre-complete algebraic type with respect to  $+$ .

- (1) We denote by  $(\sigma\text{-alg})$  the category of  $\sigma$ -algebras.
- (2) A *complete  $\sigma$ -algebra* is a  $\sigma$ -algebra  $A$  satisfying:
  - (a)  $A$  is a complete idempotent monoid, and the supremum map coincides with  $+$ , when restricted to any finite subset of  $A$ .
  - (b) Any  $n$ -ary operator  $\phi : A^n \rightarrow A$  is  $n$ -linear with respect to  $+$ :

$$\phi(x_1, \dots, \sum_j x_{ij}, \dots, x_n) = \sum_j \phi(x_1, \dots, x_{ij}, \dots, x_n).$$

- (3) A *homomorphism  $f : A \rightarrow B$  of complete  $\sigma$ -algebras* is a homomorphism of  $\sigma$ -algebras and homomorphism of complete idempotent monoids.
- (4) We denote by  $(\sigma^\dagger\text{-alg})$  the category of complete  $\sigma$ -algebras. We refer to  $\sigma^\dagger$  as a *complete algebraic type*.

We will see that the underlying functor  $U : (\sigma^\dagger\text{-alg}) \rightarrow (\sigma\text{-alg})$  has a left adjoint, but we will analyse this left adjoint more precisely, for future references.

**Definition 1.1.4.** Let  $A$  be a complete  $\sigma$ -algebra.

- (1) Let  $a$  be an element of  $A$ . A subset  $S$  of  $A$  is a *covering of  $a$* , if  $a \leq \sup S$ .

- (2) An element  $a$  of  $A$  is *compact*, if any covering of  $a$  has a finite subcovering of  $a$ .
- (3) We say that  $A$  is *algebraic*, if the following holds:
  - (a) For any element  $a$  of  $A$  is *algebraic*, i.e.  $a$  has a covering  $S$  which consists of compact elements.
  - (b) Any operator (including the infimum of finite elements)  $\phi : A^n \rightarrow A$  preserves compactness, i.e.  $\phi(x_1, \dots, x_n)$  is compact if  $x_i$ 's are compact elements.
- (4) A *homomorphism*  $f : A \rightarrow B$  of *algebraic complete  $\sigma$ -algebras* is a homomorphism of complete  $\sigma$ -algebras, sending any compact element to a compact element.
- (5) We denote by  $(\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})$  the category of algebraic complete  $\sigma$ -algebras.

These notation come from the lattice theory.

Note that any constant (regarded as a 0-ary operator) in an algebraic complete  $\sigma$ -algebra is compact, by definition. In particular, the absorbing element 1 is compact.

**Proposition 1.1.5.** Let  $\sigma$  be a pre-complete algebraic type. Then, the underlying functor  $U : (\sigma^\dagger\text{-}\mathbf{alg}) \rightarrow (\sigma\text{-}\mathbf{alg})$  has a left adjoint comp. Further, comp factors through  $(\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})$ :

$$\begin{array}{ccc}
 (\sigma\text{-}\mathbf{alg}) & \xrightarrow{\text{comp}} & (\sigma^\dagger\text{-}\mathbf{alg}) \\
 \text{comp}' \downarrow & \nearrow U' & \\
 (\mathbf{alg}.\sigma\text{-}\mathbf{alg}) & & 
 \end{array}$$

*Proof.* Let  $A$  be a  $\sigma$ -algebra. A *filter*  $F$  on  $A$  is a non-empty subset of  $A$  satisfying:

$$x, y \in F \Leftrightarrow x + y \in F.$$

We denote by  $\langle x \rangle$  the filter generated by an element  $x \in A$ . Let  $A^\dagger$  be the set of all filters on  $A$ . Given a family of filters  $\mathcal{F} = \{F_\lambda\}_\lambda$ , the supremum of  $\mathcal{F}$  is the filter generated by  $F_\lambda$ 's, i.e.  $\xi \in \sup \mathcal{F}$  if and only if there are *finite* number of  $x_\lambda \in F_\lambda$ 's such that  $\sum_\lambda x_\lambda \geq \xi$ . Then, the unit is  $\{0\}$  and the absorbing element is  $A$ . We can easily see that that 1 is compact, and any element is algebraic: a compact filter is precisely, a finitely generated filter. Let  $\phi : A^n \rightarrow A$  be another operator of  $A$ . The operator  $\phi^\dagger : (A^\dagger)^n \rightarrow A^\dagger$  associated to  $\phi$  is defined by

$$\phi^\dagger(F_1, \dots, F_n) = \sum_{x_i \in F_i} \langle \phi(x_1, \dots, x_n) \rangle.$$

This sends a  $n$ -uple of compact filters to a compact filter. This gives a structure of an algebraic complete  $\sigma$ -algebra on  $A^\dagger$ . Given a homomorphism  $f : A \rightarrow B$

of  $\sigma$ -algebras, a homomorphism  $f^\dagger : A^\dagger \rightarrow B^\dagger$  of complete  $\sigma$ -algebras is given by

$$f^\dagger(F) = \sum_{a \in F} \langle f(a) \rangle.$$

It is easy to see that  $f^\dagger$  sends a compact filter to a compact filter. Hence, we have a functor  $\underline{\text{comp}}' : (\sigma\text{-alg}) \rightarrow (\mathbf{alg}.\sigma^\dagger\text{-alg})$ . Set  $\underline{\text{comp}} = U' \circ \underline{\text{comp}}'$ , where  $U' : (\sigma^\dagger\text{-alg}) \rightarrow (\mathbf{alg}.\sigma^\dagger\text{-alg})$  is the underlying functor.

Finally, we will show that  $\underline{\text{comp}}$  is the left adjoint of  $U$ : the unit  $\epsilon : \text{Id}_{(\sigma\text{-alg})} \Rightarrow U \circ \underline{\text{comp}}$  is given by  $A \ni a \mapsto \langle a \rangle \in A^\dagger$ .  $\eta : \underline{\text{comp}} \circ U \Rightarrow \text{Id}_{(\mathbf{alg}.\sigma^\dagger\text{-alg})}$  is given by  $B^\dagger \ni F \mapsto \sup F \in B$ .  $\square$

**Remark 1.1.6.** The functor  $\underline{\text{comp}}'$  constructed above is *not* the left adjoint of the underlying functor  $(\mathbf{alg}.\sigma^\dagger\text{-alg}) \rightarrow (\sigma\text{-alg})$ : the counit  $\eta$  is not algebraic, i.e. it does not necessarily preserve compactness. However, it is an equivalence of categories: see below.

**Proposition 1.1.7.** Let  $\sigma$  be a pre-complete algebraic type. Then the above functor  $\underline{\text{comp}}'$  gives an equivalence between the category of  $\sigma$ -algebras and the category of algebraic complete  $\sigma$ -algebras.

*Proof.* We will construct a functor  $U_{\text{cpt}} : (\mathbf{alg}.\sigma^\dagger\text{-alg}) \rightarrow (\sigma\text{-alg})$  as follows: for an algebraic complete  $\sigma$ -algebra  $R$ , let  $R_{\text{cpt}}$  be the set of compact elements of  $R$ . This set has the natural induced structure of a  $\sigma$ -algebra, since all the operators are algebraic. Also, given a homomorphism  $f : A \rightarrow B$  of algebraic complete  $\sigma$ -algebras, we obtain a homomorphism  $f_{\text{cpt}} : A_{\text{cpt}} \rightarrow B_{\text{cpt}}$  of  $\sigma$ -algebras. Hence, sending  $R$  to  $R_{\text{cpt}}$  gives a functor  $U_{\text{cpt}} : (\mathbf{alg}.\sigma\text{-alg}) \rightarrow (\sigma\text{-alg})$ .

We will see that  $U_{\text{cpt}}$  is the inverse of  $\underline{\text{comp}}'$ . The unit  $\epsilon : \text{Id}_{(\sigma\text{-alg})} \Rightarrow U_{\text{cpt}} \circ \underline{\text{comp}}'$  is given by

$$A \ni a \mapsto \langle a \rangle \in (A^\dagger)_{\text{cpt}}.$$

This is an isomorphism: the inverse is given by  $F \mapsto \sup F$ . Note that this is well defined, since a compact filter is finitely generated, hence only finitely many elements is involved when taking its supremum.

The counit  $\eta : \underline{\text{comp}}' \circ U_{\text{cpt}} \Rightarrow \text{Id}_{(\mathbf{alg}.\sigma^\dagger\text{-alg})}$  is given by

$$(B_{\text{cpt}})^\dagger \ni F \mapsto \sup F \in B.$$

This is well defined, since all the filters are generated by compact elements of  $B$ , hence  $\eta$  is algebraic. The inverse of  $\eta$  is given by

$$B \ni b \mapsto \sum_{b' \leq b} \langle b' \rangle \in (B_{\text{cpt}})^\dagger,$$

where  $b'$  runs through all the compact elements smaller than  $b$ . It is clear that  $\eta^{-1}$  preserves compactness. Also,  $\eta^{-1}$  preserves any operator  $\phi$  of  $\sigma$ , since  $\phi$  preserves compactness. Hence,  $\eta^{-1}$  is well defined as a homomorphism of algebraic complete  $\sigma$ -algebras.  $\square$



**Corollary 1.1.8.** Let  $\sigma$  be a pre-complete algebraic type.

- (1) The underlying functor  $U' : (\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg}) \rightarrow (\sigma^\dagger\text{-}\mathbf{alg})$  has a right adjoint: it is  $\underline{\mathbf{alg}} = \underline{\mathbf{comp}}' \circ U$ , where  $U : (\sigma^\dagger\text{-}\mathbf{alg}) \rightarrow (\sigma\text{-}\mathbf{alg})$  is the underlying functor.
- (2) The category  $(\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})$  is small complete and small co-complete.
- (3) Let  $\tau$  be another pre-complete algebraic type stronger than  $\sigma$ . Then we have a following natural commutative diagram of functors:

$$\begin{array}{ccc} (\tau\text{-}\mathbf{alg}) & \xrightarrow{\simeq} & (\mathbf{alg}.\tau^\dagger\text{-}\mathbf{alg}) \\ \downarrow & & \downarrow \\ (\sigma\text{-}\mathbf{alg}) & \xrightarrow{\simeq} & (\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg}) \end{array}$$

where the downward arrows are the underlying functors. Hence, the underlying functor  $U : (\mathbf{alg}.\tau^\dagger\text{-}\mathbf{alg}) \rightarrow (\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})$  has a left adjoint.

**Proposition 1.1.9.** Let  $\sigma$  be a pre-complete algebraic type.

- (1) The category  $(\sigma^\dagger\text{-}\mathbf{alg})$  is small complete.
- (2) The category  $(\sigma^\dagger\text{-}\mathbf{alg})$  is small co-complete.

*Proof.* (1) We only need to verify that small products and equalizers exist, but this is just the analogue of the case of usual algebras.

- (2) We need to verify that small co-products and co-equalizers exist. First, we will verify the existence of co-equalizers. Let  $A \xrightarrow{f,g} B$  be two homomorphisms between two  $\sigma$ -algebras  $A$  and  $B$ . Let  $\mathfrak{a}$  be the *congruence relation* generated by  $f$  and  $g$ , i.e. the minimal equivalence relation on  $B$  satisfying:

- (a)  $(f(a), g(a)) \in \mathfrak{a}$  for any  $a \in A$ .
- (b) If  $(a_i, b_i) \in \mathfrak{a}$  for all  $i$ , then  $(\phi(a_1, \dots, a_n), \phi(b_1, \dots, b_n)) \in \mathfrak{a}$  for any  $n$ -ary operator  $\phi$ .
- (c) If  $(a_\lambda, b_\lambda) \in \mathfrak{a}$  for all  $\lambda$ , then  $(\sum a_\lambda, \sum b_\lambda) \in \mathfrak{a}$ .

Then, it is clear that  $B/\mathfrak{a}$  is the co-equalizer of  $f$  and  $g$ .

Next, we will show that small co-products exist. Let  $\mathcal{A} = \{A_\lambda\}_\lambda$  be a small family of complete  $\sigma$ -algebras. Let  $S$  be the epic undercategory of  $\mathcal{A}$ , i.e. its objects are pairs  $\langle B, \{f_\lambda : A_\lambda \rightarrow B\} \rangle$  satisfying:

- (a)  $B$  is a complete  $\sigma$ -algebra.
- (b)  $f_\lambda$  is a homomorphism of complete  $\sigma$ -algebras.
- (c) The image of  $f_\lambda$ 's generate  $B$  as a complete  $\sigma$ -algebra.

Then, we see that the set  $S$  is small, and complete when regarded as a category, since  $(\sigma^\dagger\text{-alg})$  is complete. The coproduct  $\amalg A_\lambda$  is then defined by the limit of  $S$ .  $\square$

**Proposition 1.1.10.** Let  $\sigma, \tau$  be two pre-complete algebraic types, and assume  $\tau$  is stronger than  $\sigma$ . Then, the underlying functor  $U_v : (\tau^\dagger\text{-alg}) \rightarrow (\sigma^\dagger\text{-alg})$  has a left adjoint.

*Proof.* Let  $F_\sigma : (\sigma\text{-alg}) \hookrightarrow (\sigma^\dagger\text{-alg}) : U_\sigma$ ,  $F_\tau : (\tau\text{-alg}) \hookrightarrow (\tau^\dagger\text{-alg}) : U_\tau$ , and  $F_\sigma : (\sigma\text{-alg}) \hookrightarrow (\tau\text{-alg}) : U_\sigma$  be adjoints, respectively:

$$\begin{array}{ccc} (\sigma\text{-alg}) & \xleftrightarrow{u} & (\tau\text{-alg}) \\ \uparrow \sigma & & \uparrow \tau \\ (\sigma^\dagger\text{-alg}) & \xleftarrow{v} & (\tau^\dagger\text{-alg}) \end{array}$$

Given a complete  $\sigma$ -algebra  $A$ , set  $A' = F_u U_\sigma A \in (\tau\text{-alg})$ . A filter  $F$  on  $A'$  is *infinite* with respect to  $A$ , if  $x_\lambda \in F \cap A$  implies  $\sum x_\lambda \in F$ . Let  $F_v(A)$  be the set of infinite filters on  $A'$ . When given a family  $\{F_\lambda\}$  of infinite filters, the supremum filter is the infinite filter generated by  $F_\lambda$ 's. The rest of the construction of the complete  $\sigma$ -algebra structure on  $F_v(A)$  is analogous to Proposition 1.1.5. Hence, we have a functor  $F_v : (\sigma^\dagger\text{-alg}) \rightarrow (\tau^\dagger\text{-alg})$ . We will show that this is the left adjoint of  $U_v$ . The unit  $\epsilon : \text{Id}_{(\sigma^\dagger\text{-alg})} \Rightarrow U_v F_v$  is given by  $A \ni a \mapsto \langle a \rangle \in F_v(A)$ . Note that this preserves the supremum map, from the advantage of using infinite filters. The counit  $\eta : F_v U_v \Rightarrow \text{Id}_{(\tau^\dagger\text{-alg})}$  is given by  $F \mapsto \sup F$ .  $\square$

## 1.2 Semirings

**Definition 1.2.1.** (1) An algebraic system  $(R, +, \times)$  is a *semiring* if:

- (a)  $R$  is a monoid with respect to  $+$  and  $\times$ .
- (b) The *distribution law* holds:

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c).$$

- (2) A semiring  $R$  is *pre-complete*, if it is pre-complete with respect to  $+$ , and if the multiplicative unit is the absorbing element with respect to  $+$ .
- (3) We denote by **(SRng)** (resp. **(PSRng)**), the category of semirings (resp. pre-complete semirings).

**Definition 1.2.2.** (1) The initial object of **(PSRng)** consists of two elements: 1 and 0. We refer to this semiring as  $\mathbb{F}_1$ , the *field with one element*. Here in the category of semirings,  $\mathbb{F}_1$  is no longer a makeshift concept.

- (2) The terminal object of  $(\mathbf{SRng})$  exists, and consists of a unique element  $1 = 0$ . We refer to this semiring as the *zero semiring*.
- (3) Let  $R$  be a complete semiring. Let  $U_m : (R/\mathbf{SRng}^\dagger) \rightarrow (\mathbf{Mnd})$  be the underlying functor, sending a complete  $R$ -algebra to its multiplicative monoid. Then,  $U_m$  has a left adjoint  $F_m$ . For a monoid  $M$ ,  $R[M] := F_m(M)$  is the *complete monoid semiring* with coefficient  $R$ . In particular, when  $M$  is the free monoid generated by a set  $S$ ,  $R[x_s]_{s \in S} = F(M)$  is the *complete polynomial semiring* with coefficient  $R$ .
- (4) Let  $A, B, R$  be complete semirings, with homomorphisms  $R \rightarrow A$  and  $R \rightarrow B$ . Then we can define the *tensor product of  $A$  and  $B$  over  $R$*  which we denote by  $A \otimes_R B$ . Of course, we have infinite tensor products too.

**Remark 1.2.3.** The constructions of various algebras are also valid in the algebraic complete case. The arguments and notations will be just the repetition, so we will skip it.

## 2 Modules over a semiring

We will construct a general fundamental theorems on  $R$ -modules, where  $R$  is a complete semiring. Note that  $(R\text{-}\mathbf{mod})$  is a little beyond from what we have been calling complete algebras: the definition of the scalar action is not algebraic, since we must consider infinite sums of elements of the coefficient ring  $R$ . However, the arguments are analogous. We just modify the definition of the filters when necessary.

### 2.1 $R$ -modules

**Definition 2.1.1.** Let  $R$  be a complete semiring.

- (1) An  $R$ -module  $M$  is a complete idempotent monoid endowed with a scalar operation  $R \times M \rightarrow M$  which satisfies the following:
  - (a)  $1 \cdot x = x$ ,  $0 \cdot x = 0$ ,  $a \cdot 0 = 0$  for any  $x \in M$  and  $a \in R$ .
  - (b) The *distribution law* holds:
$$a(\sum_\lambda x_\lambda) = \sum_\lambda (ax_\lambda) \text{ for any } a \in R, x_\lambda \in M.$$

$$(\sum_\lambda a_\lambda)x = \sum_\lambda (a_\lambda x) \text{ for any } a_\lambda \in R, x \in M.$$
  - (c)  $(ab)x = a(bx)$  for any  $a, b \in R$  and  $x \in M$ .
- (2) A map  $f : M \rightarrow N$  between two  $R$ -modules is a *homomorphism of  $R$ -modules* if it is a homomorphism of complete idempotent monoids, and preserves the scalar operation.
- (3) We denote by  $(R\text{-}\mathbf{mod})$  the category of  $R$ -modules.

**Proposition 2.1.2.** (1) The category of  $\mathbb{F}_1$ -modules coincides with the category of complete idempotent monoids.

- (2) Let  $R$  be a complete semiring. For any two  $R$ -modules  $M$  and  $N$ ,  $\text{Hom}_{(R\text{-}\mathbf{mod})}(M, N)$  has the natural structure of a  $R$ -module.

The proofs are obvious.

**Proposition 2.1.3.** Let  $R$  be a complete semiring.

- (1) The category  $(R\text{-}\mathbf{mod})$  is complete.
- (2) The category  $(R\text{-}\mathbf{mod})$  is co-complete.

*Proof.* (1) It is just the same argument of classical modules.

- (2) It suffices to show that there is small co-products and coequalizers. The construction of a coequalizer is easy. We will construct a co-product for a small family  $\{M_\lambda\}$  of  $R$ -modules. Set  $M_\infty = \coprod M_\lambda$ : the set theoretic product of  $M_\lambda$ 's. We will show that this is also the co-product. The inclusion  $M_\lambda \rightarrow M_\infty$  is given by  $x \mapsto [\mu \mapsto x\delta_{\mu,\lambda}]$  where  $\delta$  is the Kronecker delta. Given a family of homomorphisms  $f_\lambda : M_\lambda \rightarrow N$ , define  $f : M_\infty \rightarrow N$  by  $(x_\lambda)_\lambda \mapsto \sum f_\lambda(x_\lambda)$ . We see that this is the unique map.  $\square$

**Definition 2.1.4.** For a complete semiring  $R$ , set  $R'$  be the pre-complete semiring, obtained from  $R$  by forgetting the infinite sum.

- (1) A (*pre-complete*)  $R'$ -module is a pre-complete monoid  $M$  equipped with a scalar product  $R' \times M \rightarrow M$  of  $R'$ , satisfying:
  - (a)  $1 \cdot x = x$ ,  $0 \cdot x = 0$ ,  $a \cdot 0 = 0$  for any  $x \in M$ ,  $a \in R$ .
  - (b) The finite distribution law holds:
    - (i)  $a(x + y) = ax + ay$  for any  $a \in R$ ,  $x, y \in M$ .
    - (ii)  $(a + b)x = ax + bx$  for any  $a, b \in R$ ,  $x \in M$ .
  - (c)  $(ab)x = a(bx)$  for any  $a, b \in R$ ,  $x \in M$ .
- (2) A map  $f : M \rightarrow N$  between two  $R'$ -modules is a *homomorphism of  $R'$ -modules*, if it is a homomorphism of pre-complete monoids, and preserves the scalar operation.
- (3) We denote by  $(R'\text{-}\mathbf{pmod})$  the category of pre-complete  $R'$ -modules.

**Proposition 2.1.5.** The underlying functor  $U : (R\text{-}\mathbf{mod}) \rightarrow (R'\text{-}\mathbf{pmod})$  has a left adjoint.

*Proof.* The proof is analogous to that of Proposition 1.1.5. Let  $M$  be a pre-complete  $R'$ -module. A  $R$ -filter on  $M$  is a non-empty subset  $F$  of  $M$  satisfying:

- (1)  $x, y \in F \Leftrightarrow x + y \in F$ .
- (2) If  $a_\lambda x \in F$  for any  $\lambda$ , then  $(\sum a_\lambda)x \in F$ .

Let  $M^\ddagger$  be the set of  $R$ -filters on  $M$ . Then,  $M^\ddagger$  becomes a  $R$ -module. Hence, we can define a functor  $\overline{\text{comp}}^R : (R'\text{-}\mathbf{pmod}) \rightarrow (R\text{-}\mathbf{mod})$  by  $M \mapsto M^\ddagger$ . The rest is the repetition of 1.1.5.  $\square$

**Remark 2.1.6.** Note that  $\overline{\text{comp}}^R(M)$  is *not* algebraic, since a infinite sum appears in the definition of the  $R$ -filter. However, the situation is different when  $R$  is algebraic.

**Definition 2.1.7.** Let  $R$  be an algebraic complete semiring.

- (1) A  $R$ -module  $M$  is *algebraic* if:
  - (a)  $M$  is algebraic as a complete idempotent monoid.
  - (b) If  $a \in R$ ,  $x \in M$  are both compact, then so is  $ax$ .
- (2) A homomorphism  $f : M \rightarrow N$  of  $R$ -modules is *algebraic* if it preserves compactness.
- (3) We denote by  $(\mathbf{alg}.R\text{-}\mathbf{mod})$  the category of algebraic  $R$ -modules.

If  $R$  is an algebraic complete semiring and  $M$  is an algebraic  $R$ -module, then the pre-complete monoid  $M_{\text{cpt}}$  consisting of compact elements of  $M$  becomes a  $R_{\text{cpt}}$ -module. Hence we have a functor  $U_{\text{cpt}} : (\mathbf{alg}.R\text{-}\mathbf{mod}) \rightarrow (R_{\text{cpt}}\text{-}\mathbf{pmod})$ .

**Proposition 2.1.8.** The above functor  $U_{\text{cpt}}$  gives an equivalence of categories  $U_{\text{cpt}} : (\mathbf{alg}.R\text{-}\mathbf{mod}) \simeq (R_{\text{cpt}}\text{-}\mathbf{pmod})$ .

*Proof.* This is the analogue of Proposition 1.1.7. Let  $M$  be a pre-complete  $R_{\text{cpt}}$ -module and  $M^\ddagger$  be the set of filters (*not* the set of  $R$ -filters) of  $M$ . The scalar operation on  $M^\ddagger$  is given by

$$F \cdot \mathcal{X} = \sum_{a \in F, x \in \mathcal{X}} \langle ax \rangle,$$

where  $F$  (resp.  $\mathcal{X}$ ) is a filter on  $R_{\text{cpt}}$  (resp.  $M$ ). Hence, we obtain a functor  $\overline{\text{comp}}' : (R_{\text{cpt}}\text{-}\mathbf{pmod}) \rightarrow (\mathbf{alg}.R\text{-}\mathbf{mod})$ . It is easy to see that this functor gives the inverse of  $U_{\text{cpt}}$ .  $\square$

**Definition 2.1.9.** Let  $R$  be a complete semiring, and  $R'$  be the pre-complete semiring obtained from  $R$  by forgetting the infinite sum. Let  $\sigma$  be an algebraic type stronger than  $R'$ -module. We denote by  $\sigma^\ddagger$  the complete algebraic type induced by  $\sigma$ , extending the finite scalar operation to the infinite scalar operation.

**Proposition 2.1.10.** Let  $R$  be a complete semiring, and  $R'$  be the pre-complete semiring obtained from  $R$  by forgetting the infinite sum. Let  $\sigma, \tau$  be an algebraic type stronger than  $R'$ -module, and suppose  $\tau$  is stronger than  $\sigma$ . Then, there is a left adjoint functor of the underlying functor  $U_v : (\tau^\ddagger\text{-}\mathbf{alg}) \rightarrow (\sigma^\ddagger\text{-}\mathbf{alg})$ .

*Proof.* This is the analogue of Proposition 1.1.10. Let  $F_\sigma : (\sigma\text{-}\mathbf{alg}) \rightleftharpoons (\sigma^\dagger\text{-}\mathbf{alg}) : U_\sigma$ ,  $F_\tau : (\tau\text{-}\mathbf{alg}) \rightleftharpoons (\tau^\dagger\text{-}\mathbf{alg}) : U_\tau$ , and  $F_\sigma : (\sigma\text{-}\mathbf{alg}) \rightleftharpoons (\tau\text{-}\mathbf{alg}) : U_\sigma$  be adjoints, respectively:

$$\begin{array}{ccc} (\sigma\text{-}\mathbf{alg}) & \xleftarrow{u} & (\tau\text{-}\mathbf{alg}) \\ \uparrow \sigma & & \uparrow \tau \\ (\sigma^\dagger\text{-}\mathbf{alg}) & \xleftarrow{v} & (\tau^\dagger\text{-}\mathbf{alg}) \end{array}$$

Given a  $\sigma^\dagger$ -algebra  $A$ , set  $A' = F_u U_\sigma A \in (\tau\text{-}\mathbf{alg})$ . Let  $F_v(A)$  be the set of  $R$ -filters on  $A'$  which is infinite with respect to  $A$ . Then,  $F_v(A)$  becomes a  $\tau^\dagger$ -algebra, and we obtain the functor  $F_v : (\sigma^\dagger\text{-}\mathbf{alg}) \rightarrow (\tau^\dagger\text{-}\mathbf{alg})$ . This becomes the left adjoint of  $U_v$ .  $\square$

**Corollary 2.1.11.** Let  $R$  be a complete semiring. The underlying functor  $(R\text{-}\mathbf{alg}) \rightarrow (R\text{-}\mathbf{mod})$  has a left adjoint  $S$ . For any  $R$ -module  $M$ ,  $S(M)$  is the *symmetric algebra* generated by  $M$ .

## 2.2 Tensor products

Throughout this subsection, we fix a complete semiring  $R$ .

**Proposition 2.2.1.** For any  $R$ -module  $N$ , the functor  $\mathrm{Hom}_{(R\text{-}\mathbf{mod})}(N, -) : (R\text{-}\mathbf{mod}) \rightarrow (R\text{-}\mathbf{mod})$  has a left adjoint.

*Proof.* Let  $M$  be another  $R$ -module. A non-empty subset  $F$  of  $M \times N$  is a *filter* if it satisfies the followings:

- (1) Let  $x, y$  and  $r$  be elements of  $M, N$  and  $R$ , respectively. Then  $(rx, y) \in F$  if and only if  $(x, ry) \in F$ .
- (2) Let  $x_\lambda$  be elements of  $M$ , and  $y$  be an element of  $N$ . Then,  $(x_\lambda, y) \in F$  for any  $\lambda$  if and only if  $(\sum_\lambda x_\lambda, y) \in F$ .
- (3) Let  $y_\lambda$  be elements of  $N$ , and  $x$  be an element of  $M$ . Then,  $(x, y_\lambda) \in F$  for any  $\lambda$  if and only if  $(x, \sum_\lambda y_\lambda) \in F$ .

We will denote by  $\sum_\lambda x_\lambda \otimes y_\lambda$  the filter generated by  $\{(x_\lambda, y_\lambda)\}_\lambda$ . We define  $M \otimes_R N$  as the set of all filters. The supremum of  $\{F_\lambda\}$  is the filter generated by  $F_\lambda$ 's. For any scalar  $r \in R$  and any filter  $F$ , the scalar operation is defined by  $r \cdot F = \{(rx, y)\}_{(x, y) \in F}$ . Given a homomorphism  $f : M \rightarrow M'$  of  $R$ -modules, the homomorphism  $f \otimes N : M \otimes N \rightarrow M' \otimes N$  is defined by  $\sum_\lambda x_\lambda \otimes y_\lambda \mapsto \sum_\lambda f(x_\lambda) \otimes y_\lambda$ .

Thus, the functor  $- \otimes N : (R\text{-}\mathbf{mod}) \rightarrow (R\text{-}\mathbf{mod})$  is well defined. We will prove that this is the left adjoint of  $\mathrm{Hom}_{(R\text{-}\mathbf{mod})}(N, -)$ . The unit  $\epsilon : \mathrm{Id} \Rightarrow \mathrm{Hom}(N, - \otimes_R N)$  is given by  $x \mapsto [y \mapsto x \otimes y]$ . The counit  $\mathrm{Hom}(N, -) \otimes_R N \Rightarrow \mathrm{Id}$  is defined by  $f \otimes y \mapsto f(y)$ .  $\square$

**Proposition 2.2.2.** Let  $R$  be a semiring, and  $A$  be a  $R$ -algebra. Then, the underlying functor  $U : (R\text{-}\mathbf{mod}) \rightarrow (A\text{-}\mathbf{mod})$  has a left adjoint  $A \otimes_R -$ .

*Proof.* The proof is the analogy of classical algebras. Let  $M$  be a  $R$ -module. The scalar multiplication of  $A$  on  $A \otimes_R M$  is given by  $a(\sum b_\lambda \otimes x_\lambda) = \sum ab_\lambda \otimes x_\lambda$ . This gives the structure of  $A$ -module on  $A \otimes_R M$ .

The unit  $\epsilon : \text{Id}_{(R\text{-mod})} \Rightarrow U \circ (A \otimes_R -)$  is given by  $x \mapsto 1 \otimes x$ . The counit  $\eta : (A \otimes_R -) \circ U \Rightarrow \text{Id}_{(A\text{-mod})}$  is given by  $\sum a_\lambda \otimes y_\lambda \mapsto \sum a_\lambda y_\lambda$ .  $\square$

**Definition 2.2.3.** Let  $F$  be the left adjoint of the underlying functor  $U : (\mathbf{Set}) \rightarrow (R\text{-mod})$ . For any set  $S$ ,  $R^S = F(S)$  is the *free  $R$ -module generated by  $S$* .

### 3 Congruence relations

#### 3.1 Idealic semirings

**Definition 3.1.1.** (1) A semiring  $R$  is *idealistic*, if the multiplicative unit 1 is the maximal element.

(2) We denote by  $(\mathbf{IRng}^\dagger)$  the full subcategory of  $(\mathbf{SRng}^\dagger)$  consisting of complete idealic semirings.

**Definition 3.1.2.** Let  $f : A \rightarrow B$  a homomorphism of complete idealic semirings. For an element  $b$  of  $B$ ,  $\pi^{-1}(b) = \sup\{x \in A \mid \pi(x) \leq b\}$  is the *inverse image* of  $b$ . Note that  $\pi^{-1}$  neither preserves supremums nor multiplications.

#### 3.2 Congruence relations

**Definition 3.2.1.** Let  $R$  be a complete semiring. A *congruence relation*  $\mathfrak{a}$  of  $R$  is an equivalence relation on  $R$  satisfying the following conditions:

- (1) If  $(a_\lambda, b_\lambda) \in \mathfrak{a}$ , then  $(\sum a_\lambda, \sum b_\lambda) \in \mathfrak{a}$ .
- (2) If  $(a_i, b_i) \in \mathfrak{a}$  for  $i = 1, 2$ , then  $(a_1 a_2, b_1 b_2) \in \mathfrak{a}$ .

If  $\mathfrak{a}$  is a congruence relation of  $R$ , then  $R/\mathfrak{a}$  has a natural structure of a complete semiring, and there is a surjective homomorphism  $\pi : R \rightarrow R/\mathfrak{a}$  of complete semirings. We denote by  $\tilde{R}$  the set of congruence relations of  $R$ .

The next proposition is obvious.

**Proposition 3.2.2.** Let  $R$  be a complete semiring. Then  $\tilde{R}$  parametrizes surjective homomorphisms of complete semirings from  $R$ , i.e. for any surjective homomorphism  $f : R \rightarrow A$  of complete semirings, there exists a unique congruence relation  $\mathfrak{a}$  of  $A$  and a natural isomorphism  $R/\mathfrak{a} \simeq A$  making the following diagram commutative:

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ \downarrow & \nearrow \simeq & \\ R/\mathfrak{a} & & \end{array}$$

**Proposition 3.2.3.** The set  $\tilde{R}$  has a natural structure of an idealic  $R$ -algebra.

*Proof.* Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be two congruence relations of  $R$ . We define the multiplication  $\mathfrak{a} \cdot \mathfrak{b}$  as the congruence relation generated by  $\{(ab + a'b', ab' + a'b)\}$ , where  $(a, a') \in \mathfrak{a}$  and  $(b, b') \in \mathfrak{b}$ . The supremum of congruence relations is the congruence relation generated by those. We can easily verify that this gives the structure of idealic semiring on  $\tilde{R}$ : The multiplicative unit is  $1_{\tilde{R}} = R \times R$ , and the additive unit is  $0_{\tilde{R}} = \Delta$ , the diagonal subset of  $R \times R$ . Finally, the homomorphism  $R \rightarrow \tilde{R}$  of semirings is given by  $a \mapsto \langle(a, 0)\rangle$ , where  $\langle(a, 0)\rangle$  is the congruence relation generated by  $(a, 0)$ .  $\square$

**Remark 3.2.4.** We must notice that a localization of a idealic semiring (to be appeared in 4.2) is also surjective, hence in the category of idealic semirings we cannot distinguish localizations from quotient rings, using only congruence relations.

**Definition 3.2.5.** Let  $R$  be a complete semiring,  $\mathfrak{a}$  be a congruence relation on  $R$ , and  $\pi : R \rightarrow R/\mathfrak{a}$  be a natural map. For an element  $a$  of  $R$ ,  $\bar{a} = \pi^{-1}\pi(a)$  is the  $\mathfrak{a}$ -saturation of  $a$ .  $a$  is  $\mathfrak{a}$ -saturated if  $\bar{a} = a$ .

**Lemma 3.2.6.** Let  $R$  be a complete idealic semiring, and  $\mathfrak{a}$  be a congruence relation on  $R$ .

- (1) For any  $a_\lambda \in R$ ,  $\overline{\sum a_\lambda} = \sum \overline{a_\lambda}$ .
- (2) For any  $a, b \in R$ ,  $\overline{\bar{a} \cdot \bar{b}} = \overline{ab}$ .

*Proof.* (1) Note that  $\pi(\pi^{-1}(x)) = x$  for any  $x \in R/\mathfrak{a}$ , since  $\pi$  is surjective.

$$\begin{aligned} \overline{\sum a_\lambda} &= \pi^{-1}(\pi(\sum \overline{a_\lambda})) = \pi^{-1}(\sum \pi(\overline{a_\lambda})) \\ &= \pi^{-1}(\sum \pi(a_\lambda)) = \pi^{-1}(\pi(\sum a_\lambda)) = \overline{\sum a_\lambda} \end{aligned}$$

- (2) Similar argument.  $\square$

**Lemma 3.2.7.** Let  $R$  be an algebraic complete semiring, and  $\mathfrak{a}$  be a congruence relation of  $\mathfrak{a}$ . Then, the following are equivalent:

- (1)  $R/\mathfrak{a}$  is algebraic, and the natural map  $\pi : R \rightarrow R/\mathfrak{a}$  is algebraic (When this happens, we call  $\mathfrak{a}$  *algebraic*.)
- (2) Let  $a \in R$  be compact, and  $b_\lambda \in R$ . Then,  $(a + \sum b_\lambda, \sum b_\lambda) \in \mathfrak{a}$  implies  $(a + \sum^{<\infty} b_\lambda, \sum^{<\infty} b_\lambda) \in \mathfrak{a}$ .

The proof is straightforward.

**Definition 3.2.8.** Let  $R$  be a complete semiring. A semiorder  $\prec$  on  $R$  is a *complete* (resp. *finite*) *idealic semiorder* if:

- (1)  $a_\lambda \prec b_\lambda$  for any  $\lambda \Rightarrow \sum a_\lambda \prec \sum b_\lambda$  (resp.  $a_1 + a_2 \prec b_1 + b_2$ ).



$$(2) \ a \leq b \Rightarrow a \prec b.$$

$$(3) \ a_i \prec b_i \ (i = 1, 2) \Rightarrow a_1 a_2 \prec b_1 b_2.$$

If  $\prec$  is a complete idealic semiorder, then the equivalence relation  $\mathfrak{a}$  defined by  $(a, b) \in \mathfrak{a} \Leftrightarrow a \prec b, b \prec a$  becomes a congruence relation on  $R$ .

**Proposition 3.2.9.** Let  $R$  be an algebraic complete semiring. Let  $\prec^f$  be a finite idealic semiorder satisfying: (\*) if  $x$  is compact and  $x \prec^f b$ , then there is a compact  $b' \leq b$  such that  $x \prec^f b'$ .

Let  $\prec$  be the complete idealic semiorder generated by  $\prec^f$ . Then, the following are equivalent:

$$(i) \ a \prec b$$

$$(ii) \ x \prec^f b \text{ holds for any compact } x \leq a.$$

Furthermore, if  $\mathfrak{a}$  is a congruence relation generated by  $\prec$ , then  $\mathfrak{a}$  is algebraic, and  $a \leq b$  in  $R/\mathfrak{a}$  if and only if  $a \prec b$ .

*Proof.* (i)  $\Rightarrow$  (ii) is clear. We will show (ii)  $\Rightarrow$  (i). Let  $\ll$  be the relation defined by  $a \ll b$  if the condition (ii) holds. It suffices to show that  $\ll$  is actually a complete semiorder. Firstly,  $\ll$  is a semiorder: indeed, if  $a \ll b$  and  $b \ll c$ , then  $x \prec^f b$  holds for any compact  $x \leq a$ . The given condition (\*) implies that there is a compact  $b' \leq b$  such that  $x \prec b'$ .  $b \ll c$  implies that  $b' \prec^f c$ , hence the result follows. It is easy to see that  $\ll$  is finite idealic. It remains to show that  $\ll$  is complete. Suppose  $a_\lambda \ll b_\lambda$ . Then, for any compact  $x \leq \sum a_\lambda$ ,

$$x \leq \sum_{<\infty} a_\lambda \prec^f \sum_{<\infty} b_\lambda \prec^f \sum b_\lambda,$$

hence  $\sum a_\lambda \ll \sum b_\lambda$ . The rest are straightforward.  $\square$

## 4 Topological spaces

In this section, we will see that the sober spaces are the most appropriate for considering spectrum functors. However, most topics in this section has been already done decades ago in the lattice theory.

### 4.1 The spectrum functor

In the sequel, all semirings are complete.

**Definition 4.1.1.** (1) We denote by **(Top)**, the category of topological spaces.

- (2) A topological space  $X$  is *sober*, if any (nonempty) irreducible closed subset of  $X$  has a unique generic point. We denote by **(Sob)** the full subcategory of **(Top)** consisting of sober topological spaces.

**Proposition 4.1.2.** The category  $(\mathbf{Sob})$  is a full co-reflective subcategory of  $(\mathbf{Top})$ , i.e. the underlying functor  $U : (\mathbf{Sob}) \rightarrow (\mathbf{Top})$  has a left adjoint.

*Proof.* This proposition is standard.

For a given topological space  $X$ , Let  $\underline{\mathbf{sob}}(X)$  be the set of irreducible closed subsets of  $X$ . Closed sets of  $\underline{\mathbf{sob}}(X)$  are of forms  $V(z) = \{c \in \underline{\mathbf{sob}}(X) \mid c \subset z\}$ , where  $z$  is a closed subset of  $X$ . Given a continuous map  $f : X \rightarrow Y$  between topological spaces,  $\underline{\mathbf{sob}}(f) : \underline{\mathbf{sob}}(X) \rightarrow \underline{\mathbf{sob}}(Y)$  is defined by  $c \mapsto \overline{f(c)}$ , where  $\overline{f(c)}$  is the closure of  $f(c)$ . This gives a functor  $\underline{\mathbf{sob}} : (\mathbf{Top}) \rightarrow (\mathbf{Sob})$ . The unit  $\epsilon : \text{Id}_{(\mathbf{Top})} \Rightarrow U \circ \underline{\mathbf{sob}}$  is given by  $x \mapsto \overline{\{x\}}$ . The counit  $\eta : \underline{\mathbf{sob}} \circ U \Rightarrow \text{Id}_{(\mathbf{Sob})}$  is given by  $z \mapsto \xi_z$ , where  $\xi_z$  is the unique generic point of  $z$ .  $\square$

**Remark 4.1.3.** The counit  $\eta$  in the above proof is actually an isomorphism: the inverse is the unit  $\epsilon$ .

**Definition 4.1.4.** (1) Given a sober space  $X$ , the set  $C(X)$  of closed sets of  $X$  becomes a complete idealic semiring with idempotent multiplication: namely, the supremum of closed sets are their intersections, and the multiplication of two closed sets is the union of the two.

(2) Given a continuous map  $f : Y \rightarrow X$  between two sober spaces, we can define  $C(f) : C(X) \rightarrow C(Y)$  by  $Z \mapsto f^{-1}(Z)$ . Thus, we obtain a contravariant functor  $C : (\mathbf{Sob})^{\text{op}} \rightarrow (\mathbf{IRng}^{\dagger})$ .

**Definition 4.1.5.** Let  $R$  be an idealic semiring.

- (1) An element  $p$  of  $R$  is *prime*, if the subset  $\{a \in R \mid a \not\leq p\}$  is a multiplicative monoid.
- (2) The *spectrum*  $\text{Spec } R$  of  $R$  is the set of all prime elements of  $R$ . For any  $a \in R$ , define  $V(a) \subset \text{Spec } R$  as the subset consisting of all prime greater than  $a$ :  $V(a) = \sup\{p \in \text{Spec } R \mid a \leq p\}$ . The subset of the form  $V(a)$  satisfies the axiom of closed sets, hence gives a topology on  $\text{Spec } R$ . It is easy to see that  $\text{Spec } R$  is a sober space.
- (3) Let  $f : A \rightarrow B$  be a homomorphism of idealic semirings. We define a continuous map  $\text{Spec } f : \text{Spec } B \rightarrow \text{Spec } A$  by

$$y \mapsto f^{-1}(y) = \sup\{x \in R \mid f(x) \leq y\}.$$

Hence, we obtain a contravariant functor  $\text{Spec} : (\mathbf{IRng}^{\dagger}) \rightarrow (\mathbf{Sob})^{\text{op}}$ , to which we refer as the *spectrum functor*.

Note that  $\text{Spec } f$  is well defined. Indeed, let  $p \in B$  be a prime element. Let  $a, b \in A$  be two elements satisfying  $a, b \not\leq f^{-1}(p)$ . This means that  $f(a), f(b) \not\leq p$ , hence  $f(ab) \not\leq p$  since  $p$  is prime. Thus we obtained  $ab \not\leq f^{-1}(p)$ .

**Proposition 4.1.6.** The spectrum functor  $\text{Spec}$  is the left adjoint of  $C : (\mathbf{Sob})^{\text{op}} \rightarrow (\mathbf{IRng}^{\dagger})$ .

*Proof.* The unit  $\epsilon : \text{Id}_{(\mathbf{IRng}^\dagger)} \Rightarrow C \circ \text{Spec}$  is given by  $A \ni a \mapsto V(a)$ . The counit  $\eta : \text{Spec} \circ C \Rightarrow \text{Id}_{(\mathbf{Sob})^{\text{op}}}$  is given by

$$X \ni x \mapsto \overline{\{x\}} \in \text{Spec} \circ C(X).$$

□

**Remark 4.1.7.** The above  $\eta$  is actually a homeomorphism: the inverse is given by sending an irreducible closed set  $Z$  to its unique generic point. This shows that  $(\mathbf{Sob})^{\text{op}}$  is a reflective full subcategory of  $(\mathbf{IRng}^\dagger)$ .

**Definition 4.1.8.** (1) Let  $X$  be a sober space. We call  $X$  *algebraic* if:

- (a)  $X$  is quasi-compact.
  - (b)  $X$  is quasi-separated, i.e. for any two quasi-compact open subsets  $U, V$  of  $X$ ,  $U \cap V$  is quasi-compact (cf. [EGA], Chap I. 1.2).
  - (c) Any open subset of  $X$  is a union of some quasi-compact open subsets.
- (2) A *morphism*  $f : X \rightarrow Y$  of algebraic sober spaces is a continuous map which is *quasi-compact*: the inverse image of a quasi-compact open subset is again quasi-compact.
- (3) We denote by **(alg.Sob)** the category of algebraic sober spaces.

A sober space  $X$  is algebraic if and only if  $C(X)$  is algebraic. An algebraic sober space is a quasi-compact coherent space, and vice versa.

**Proposition 4.1.9.** Let  $R$  be an algebraic complete idealic semiring, and  $X = \text{Spec } R$ . Denote by  $D(a)$  the open subset  $X \setminus V(a)$  for any element  $a \in R$ .

- (1) Let  $U$  be an open subset of  $X$ . Then,  $U$  is quasi-compact if and only if there is a compact element  $a$  of  $R$  such that  $U = D(a)$ .
- (2)  $X$  is algebraic.
- (3) If  $\varphi : B \rightarrow A$  is a homomorphism of algebraic complete idealic semirings, then  $\text{Spec}(\varphi) : \text{Spec } B \rightarrow \text{Spec } A$  is algebraic.

*Proof.* (1) Suppose  $U$  is quasi-compact. There is an element  $b \in R$  such that  $U = D(b)$ . There is a covering  $b = \sum b_\lambda$  of  $b$  by compact elements. Then  $U = \cup D(b_\lambda)$ , and since  $U$  is quasi-compact, there is a finite subcover  $U = \cup^{<\infty} D(b_\lambda)$ . This means that  $U = D(\sum^{<\infty} b_\lambda)$ , and  $\sum^{<\infty} b_\lambda$  is compact. The converse is easy.

- (2) First, quasi-compactness of  $X$  follows from (1) since the unit 1 is compact. Let  $U_1, U_2$  be two quasi-compact open subsets of  $X$ . Then, there are compact elements  $a_i \in R$  such that  $U_i = D(a_i)$ . Then  $a_1 a_2$  is compact, so  $U_1 \cap U_2 = D(a_1 a_2)$  is quasi-compact. Thus,  $X$  is quasi-separated. Any open subset of  $X$  is a union of quasi-open subsets, since any element of  $R$  is algebraic. Thus,  $X$  is algebraic.

(3) Easy. □

**Corollary 4.1.10.** The spectrum functor gives a functor  $\text{Spec} : (\mathbf{alg. IRng}^\dagger) \rightarrow (\mathbf{alg. Sob})^{\text{op}}$  when restricted to  $(\mathbf{alg. IRng}^\dagger)$ , and is the left adjoint of  $C : (\mathbf{alg. Sob})^{\text{op}} \rightarrow (\mathbf{alg. IRng}^\dagger)$ .

*Proof.* To see the adjointness, it suffices to show that the unit  $\epsilon$  and the counit  $\eta$  in Proposition 4.1.6 are algebraic. The above proposition shows that  $\epsilon$  is algebraic, and  $\eta$  is algebraic since it is a homeomorphism. □

We will strengthen this result later. (See Proposition 4.3.2.)

## 4.2 Localization

In this subsection, we gather some basic facts on localization of idealic semirings. Most of the results are similar to those of ordinary rings.

**Definition 4.2.1.** Let  $R$  be a complete idealic semiring.

- (1) A subset  $\Sigma$  of  $R$  is a *multiplicative system* of  $R$  if it is a submonoid of  $R$ .  $\Sigma$  is *compact*, if it consists of compact elements.
- (2) The *localization of  $R$  along  $\Sigma$*  is the homomorphism  $R \rightarrow R/\mathfrak{a}$  of complete idealic semirings, where  $\mathfrak{a}$  is a congruence relation on  $R$  generated by pairs  $(1, s)$ , where  $s \in \Sigma$ . We write  $\Sigma^{-1}R$  instead of  $R/\mathfrak{a}$ .
- (3) If  $\Sigma_f = \{f^n\}_{n \in \mathbb{N}}$  for an element  $f \in R$ , we set  $R_f = \Sigma_f^{-1}R$ .
- (4) If  $\Sigma_p = \{x \mid x \not\leq p\}$  for a prime element  $p$  of  $R$ , we set  $R_p = \Sigma_p^{-1}R$ , and refer to this semiring the *localization of  $R$  along  $p$* . If  $R$  is algebraic, then we set  $\Sigma_p = \{x \mid x \text{ is compact and } x \not\leq p\}$  instead.
- (5) Given an element  $a$  of  $R$ ,  $\bar{a} = \pi^{-1}\pi(a)$  is the  $\Sigma$ -saturation of  $a$ , where  $\pi : R \rightarrow \Sigma^{-1}R$  is the natural map.  $a$  is  $\Sigma$ -saturated if  $\bar{a} = a$ .

**Lemma 4.2.2.** Let  $R$  be an algebraic complete idealic semiring, and  $\Sigma$  be a compact multiplicative system. Let  $\mathfrak{a}$  be the congruence relation defined by  $\Sigma$ . Then  $(f, g) \in \mathfrak{a}$  if and only if:

- (b) For any compact  $x$  smaller than  $f$ , there exists  $s \in \Sigma$  such that  $sx \leq g$ . For any compact  $y$  smaller than  $g$ , there exists  $t \in \Sigma$  such that  $ty \leq f$ .

*Proof.* We make use of Proposition 3.2.9. Let  $\prec^f$  be a relation defined by  $a \prec^f b$  if:

For any compact  $x \leq a$ , there exists  $s \in \Sigma$  such that  $sx \leq b$ .

We can easily see that  $\prec^f$  is a finite idealic semiorder, and  $\mathfrak{a}$  is the congruence relation generated by  $\prec^f$ . It remains to show that if  $x \prec^f b$  is compact, then there exists a compact  $b' \leq b$  such that  $x \prec^f b'$ . Since  $sx \leq b$  for some  $s \in \Sigma$ , and  $sx$  is compact, there exists a compact  $b' \leq b$  satisfying  $sx \leq b'$ . Hence, the result follows. □

**Corollary 4.2.3.** Let  $R$  be an algebraic complete idealic semiring, and  $\Sigma$  be a compact multiplicative system. Then:

- (1)  $\Sigma^{-1}R$  is algebraic and the natural map  $\pi : R \rightarrow \Sigma^{-1}R$  is also algebraic.
- (2) More generally, let  $\Sigma_1, \Sigma_2$  be compact multiplicative systems. If there is a homomorphism  $f : \Sigma_1^{-1}R \rightarrow \Sigma_2^{-1}R$  of  $R$ -algebras, then  $f$  is algebraic.
- (3) For any  $a \in R$ ,  $\bar{a}$  is the supremum of all compact  $x \in R$  satisfying  $sx \leq a$  for some  $s \in \Sigma$ .

*Proof.* These are direct consequences of Lemma 3.2.6 and Lemma 4.2.2. Let us prove (2). Suppose  $x \in \Sigma_2^{-1}R$  is a compact element. Then, we can find a compact  $a \in R$  such that  $\pi_2(a) = x$ , where  $\pi_i : R \rightarrow \Sigma_i^{-1}R$  is the natural map:

$$\begin{array}{ccc} R & & \\ \pi_1 \downarrow & \searrow \pi_2 & \\ \Sigma_1^{-1}R & \xrightarrow{f} & \Sigma_2^{-1}R \end{array}$$

Then,  $f(x) = \pi_2(a)$ , hence compact. □

**Corollary 4.2.4.** Let  $R$  be an algebraic complete idealic semiring.

- (1) Let  $f$  be a compact element in  $R$ . The localization map  $R \rightarrow R_f$  induces  $\text{Spec } R_f \rightarrow \text{Spec } R$ , which is an open immersion:  $\text{Spec } R_f \simeq D(f) = \text{Spec } R \setminus V(f)$ .
- (2) The open sets of the form  $D(f)$  with  $f$  compact gives an open basis of  $\text{Spec } R$ .

**Corollary 4.2.5.** Let  $R$  be an algebraic complete idealic semiring, and  $p$  a prime element of  $R$ . Then,  $R_p$  is an algebraic complete idealic local semiring, with  $pR_p$  being the unique maximal ideal. Here,  $pR_p$  is the image of  $p$  via the natural map  $\pi : R \rightarrow R_p$ .

**Lemma 4.2.6.** Let  $R$  be a complete idealic semiring.

- (1) Any maximal element (of  $R \setminus \{1\}$ ) is prime.
- (2) Suppose  $R$  is algebraic. Then, for any non-unit element  $a \neq 1$ , there exists a maximal element larger than  $a$ .

*Proof.* (1) Let  $m$  be a maximal element of  $R$ , and suppose  $a \not\leq m$  and  $b \not\leq m$ . then  $a + m = b + m = 1$  since  $m$  is maximal. Hence,  $m + ab \geq (m + a)(m + b) = 1$  which shows that  $ab \leq m$ . Thus,  $m$  is prime.

- (2) Let  $\mathcal{S}$  be a set of non-unit elements which are larger than  $a$ . Then  $\mathcal{S}$  is a inductively ordered set, since 1 is compact. Hence, there is a maximal element of  $\mathcal{S}$  by Zorn's lemma. □

**Lemma 4.2.7.** Let  $R$  be an algebraic complete idealic ring, and  $a, b \in R$ . Then, the following are equivalent:

- (i) For any compact  $x \leq a$ , there exists a natural number  $n$  such that  $x^n \leq b$ .
- (ii)  $\sqrt{a} \leq \sqrt{b}$ , where  $\sqrt{a} = \sup\{x \mid x^n \leq a \text{ for some } n\}$ .
- (iii)  $V(a) \supset V(b)$  on  $\text{Spec } R$ .

Further, if  $a$  is compact, these are also equivalent to:

- (iv)  $\mathfrak{a} \geq \mathfrak{b}$ , where  $\mathfrak{a}$  (resp.  $\mathfrak{b}$ ) is a congruence relation generated by  $(1, a)$  (resp.  $(1, b)$ ).

*Proof.* (i)  $\Rightarrow$  (ii): If  $x \leq \sqrt{a}$  is any compact element, then there exists a natural number  $m$  such that  $x^m \leq a$ . Then, (i) implies that  $x^{mn} \leq b$  for some  $n$ , hence  $x \leq \sqrt{b}$ . This shows that  $\sqrt{a} \leq \sqrt{b}$ .

(ii)  $\Rightarrow$  (iii): If  $p \in V(b)$ , then  $p \geq b$ . Let  $x \leq a$  be any compact element. Then  $x \leq \sqrt{a} \leq \sqrt{b}$  implies  $x^m \leq b$  for some  $m$ . Since  $x^m \leq p$  and  $p$  is prime, we have  $x \leq p$ . Therefore,  $a \leq p$  and  $p \in V(a)$ .

(iii)  $\Rightarrow$  (i): Suppose there exists a compact  $x \leq a$  such that  $x^n \not\leq b$  for any  $n$ . Set  $A = R/b$ : this is the quotient ring of  $R$  divided by a congruence relation generated by  $(b, 0)$ . Then  $x^n \not\leq 0$  for all  $n$  in  $A$ . This shows that  $A_x \neq 0$ . Since  $A_x$  is algebraic, we can find a prime element  $p$  of  $A_x$ , by Lemma 4.2.6. Let  $\varphi : R \rightarrow A_x$  be the canonical map. Then,  $\varphi^{-1}(p)$  is contained in  $V(b)$ , but  $\varphi^{-1}(p) \notin V(a)$ , since  $\varphi^{-1}(p) \not\leq x$ . This is a contradiction.

Suppose  $a$  is compact.

(i)  $\Rightarrow$  (iv):  $a^n \leq b$  holds for some  $n$ , and since  $(a^n, 1) \in \mathfrak{a}$ , we have  $(b, 1) \in \mathfrak{a}$ .

(iv)  $\Rightarrow$  (i):  $(1, b) \in \mathfrak{b}$ , so  $(1, b) \in \mathfrak{a}$  by assumption. This is equivalent to  $a^n \leq b$  for some  $n$ .  $\square$

**Corollary 4.2.8.** Let  $R$  be an algebraic complete idealic semiring, and  $U$  be a quasi-compact open subset of  $X = \text{Spec } R$ . Set  $Z = X \setminus U$ . Let  $f, g \in R$  be compact elements satisfying  $V(f) = V(g) = Z$ . Then  $R_f \simeq R_g$ .

**Definition 4.2.9.** Let  $R$  be an algebraic complete idealic semiring. By the above corollary, we may define a *localization*  $R_Z$  of  $R$  along any compact  $Z \in C(X)$ , by  $R_Z = R_f$  for any compact  $f \in R$  such that  $V(f) = Z$ . Also, for a non-compact  $Z \in C(X)$ , set  $R_{[Z]} = \varprojlim_{Z' \leq Z} R_{Z'}$ , where  $Z'$  runs through all the compact elements smaller than  $Z$ , and the limit is taken within the category of algebraic semirings.

**Remark 4.2.10.** Note that  $R_{[Z]}$  is not isomorphic to  $R_Z$  in general: if  $Z$  is not compact,  $R_Z$  may not be even algebraic.

The next is the key lemma, which is indispensable when constructing idealic schemes.

**Lemma 4.2.11.** Let  $R$  be an algebraic complete idealic semiring, and  $s, s_1, \dots, s_n$  be compact elements satisfying  $s = \sum s_i$ . If there are elements  $f_i \in R_i = R_{s_i}$  satisfying  $f_i = f_j$  in  $R_{ij} = R_{s_i s_j}$ , then there is a unique element  $f \in R_s$  such that  $f = f_i$  in  $R_{s_i}$ .

*Proof.* First, we will show the uniqueness of  $f$ . Let  $f$  and  $g$  be elements of  $R_s$ , such that  $f = f_i = g$  in  $R_i$ . For any compact  $x \leq f$ , there exist natural numbers  $m_i$  such that  $s_i^{m_i} x \leq g$  in  $R$  for any  $i$ . Set  $m = \max_i m_i$ . Then,  $\sum s_i = s$  implies

$$\sum s_i^{m_i} \geq \sum s_i^m \geq (\sum s_i)^{nm} = s^{nm}$$

so that  $s^{nm} x \leq g$ . This means that  $x \leq g$  in  $R_s$ , so  $f \leq g$ .  $g \leq f$  can be shown in a similar way.

Next, we will show the existence. Set  $f = \sum f_i$ . To show  $f = f_i$  in  $R_i$ , it suffices to prove  $f_j \leq f_i$  in  $R_i$ . Since  $f_i = f_j$  in  $R_{ij}$ , there exists natural numbers  $m_{ij}$  such that  $(s_i s_j)^{m_{ij}} f_j \leq f_i$  for any  $i, j$ . Set  $m = \max_{i,j} m_{ij}$ . Then

$$\sum_j (s_i s_j)^{m_{ij}} \geq \sum_j (s_i s_j)^m \geq s_i^m (\sum_j s_j)^{mn} = s_i^m s^{mn}$$

so that  $s_i^m s^{mn} f_j \leq f_i$ . Since  $s_i \leq s$ , this means that  $s_i^{m(n+1)} f_j \leq f_i$ , hence  $f_j \leq f_i$  in  $R_i$ .  $\square$

**Remark 4.2.12.** Note that this lemma also holds for pre-complete idealic semirings.

### 4.3 Comparison with the Stone-Čech compactification

**Definition 4.3.1.** We denote by  $(\mathbf{alg.IIRng}^\dagger)$  the full subcategory of  $(\mathbf{alg.IRng}^\dagger)$  consisting of algebraic complete idealic semirings with idempotent multiplications.

**Proposition 4.3.2.** The spectrum functor and the functor  $C$  introduced in 4.1.10 gives an equivalence between the category  $(\mathbf{alg.IIRng}^\dagger)$  and  $(\mathbf{alg.Sob})^{\text{op}}$ , the (opposite) category of algebraic sober spaces.

*Proof.* We only need to prove that the unit  $\epsilon : \text{Id}_{(\mathbf{alg.IIRng}^\dagger)} \Rightarrow C \circ \text{Spec}$  of Corollary 4.1.10 is a natural isomorphism. Let  $R$  be an algebraic complete idealic semiring, and  $a, b \in R$ . It suffices to show that  $V(a) \leq V(b)$  implies  $a \leq b$ . If  $V(a) \leq V(b)$ , then Lemma 4.2.7 says that for any compact  $x$ ,  $x^n \leq b$  for some  $b$ , but since the multiplication is idempotent, this means that  $x \leq b$ , hence  $a \leq b$ .  $\square$

Making use of Corollary 1.1.8, we have the following:

**Proposition 4.3.3.** The underlying functor  $U : (\mathbf{alg.Sob}) \rightarrow (\mathbf{Sob})$  has a left adjoint  $\underline{\text{alg}}$ .

*Proof.* The functor  $\underline{\text{alg}}$  preserves the idempotency of multiplication,  $(\mathbf{Sob})$  is a subcategory of  $(\mathbf{IIRng}^\dagger)$ , and  $(\mathbf{alg.Sob})$  coincides with  $(\mathbf{alg.IIRng}^\dagger)$ , which shows that  $\underline{\text{alg}}$  sends a sober space to an algebraic sober space.

This is enough for a proof, but let us write it down this functor explicitly, for future references.

Given a sober space  $X$ , let  $\underline{\text{alg}}(X)$  be the set of prime filters on  $C(X)$ : a prime filter  $F$  is a non-empty subset of  $C(X)$  satisfying:

$$(1) \ C_1, C_2 \in F \Leftrightarrow C_1 \cap C_2 \in F.$$

$$(2) \ C_1, C_2 \notin F \Rightarrow C_1 \cup C_2 \notin F.$$

A closed set of  $\underline{\text{alg}}(X)$  is of a form  $V(a) = \{F \mid a \subset F\}$ , where  $a$  is a filter on  $C(X)$ . Given a continuous map  $f : X \rightarrow Y$  of sober spaces, a continuous map  $\underline{\text{alg}}(f) : \underline{\text{alg}}(X) \rightarrow \underline{\text{alg}}(Y)$  is given by

$$\underline{\text{alg}}(X) \ni p \mapsto \sum_{f^{-1}(z) \in p} \langle z \rangle \in \underline{\text{alg}}(Y),$$

where  $z$  runs through all the closed subsets of  $Y$ , satisfying  $f^{-1}(z) \in p$ . Thus, we have a functor  $\underline{\text{alg}} : (\mathbf{Sob}) \rightarrow (\mathbf{alg.Sob})$ . The unit  $\epsilon : \text{Id}_{(\mathbf{Sob})} \Rightarrow U \underline{\text{alg}}$  of the adjoint is given by  $x \mapsto \langle \overline{\{x\}} \rangle$ , and the counit  $\eta : \underline{\text{alg}} U \Rightarrow \text{Id}_{(\mathbf{alg.Sob})}$  is given by  $p \mapsto \bigcap_{c \in p} c$ .  $\square$

**Proposition 4.3.4.** If  $X \in (\mathbf{Top})$  is a Hausdorff space, then  $\underline{\text{alg}}(X)$  is also Hausdorff.

*Proof.* The topological space  $X$  is Hausdorff if and only if the diagonal functor  $\Delta : X \rightarrow X \times X$  is a closed immersion. It is obvious that the functor  $\underline{\text{alg}}$  preserves closed immersion, hence  $\underline{\text{alg}}(X) \rightarrow \underline{\text{alg}}(X) \times \underline{\text{alg}}(X)$  is also a closed immersion.  $\square$

**Definition 4.3.5.** Hausdorff algebraic sober spaces are called *Stone spaces*. The category of Stone spaces and continuous maps is denoted by  $(\mathbf{Stone})$ .

**Remark 4.3.6.** Note that a continuous map between Stone spaces are already quasi-compact: an open subset  $U$  of a Stone space  $X$  is quasi-compact if and only iff  $U$  is clopen, since  $X$  is Hausdorff.

**Definition 4.3.7.** An II-Ring  $R$  is a *Boolean algebra*, if there is a unary operator  $\neg : R \rightarrow R$  such that

$$a + \neg a = 1, \quad a \cdot \neg a = 0.$$

**Proposition 4.3.8.** (1) The category  $(\mathbf{Bool})$  of Boolean algebras is equivalent to the opposite category of  $(\mathbf{Stone})$ .

(2) The underlying functor  $(\mathbf{Stone}) \rightarrow (\mathbf{alg.Sob})$  has a left adjoint and a right adjoint.

*Proof.* (1) This is obvious from the above remark.



- (2) The existence of the right adjoint follows from the fact that we have a left adjoint of the underlying functor  $(\mathbf{alg.IIRng}^\dagger) \rightarrow (\mathbf{Bool})$ .

We will construct the left adjoint. Let  $X$  be an algebraic sober space, and  $A = C(X)_{\text{cpt}}$  be the corresponding pre-complete idealic semiring. Let  $B$  be the subring of  $A$  consisting of elements with *negation*, namely elements  $x \in A$  which satisfies  $xy = 0$  and  $x + y = 1$  for some  $y \in A$ . Note that  $y$  is uniquely determined by  $x$ . Therefore, we see that  $B$  becomes a Boolean algebra. This correspondence gives a functor  $G : (\mathbf{alg.IIRng}^\dagger) \rightarrow (\mathbf{Bool})$ , and hence  $G^{\text{op}} : (\mathbf{alg.Sob}) \rightarrow (\mathbf{Stone})$ . For any Boolean algebra  $C$ , any morphism  $C \rightarrow A$  of pre-complete idempotent semirings factors through  $C \rightarrow B$ . This shows that  $G^{\text{op}}$  is the left adjoint of the underlying functor.  $\square$

**Remark 4.3.9.** Recall the Stone-Čech compactification:

**Theorem 4.3.10.** Let  $U : (\mathbf{CptHaus}) \rightarrow (\mathbf{Haus})$  be the underlying functor from the category  $(\mathbf{CptHaus})$  of compact Hausdorff spaces to the category  $(\mathbf{Haus})$  of Hausdorff spaces. Then,  $U$  has a left adjoint  $\beta$ . The unit morphism  $X \rightarrow \beta X$  is an open immersion if and only if  $X$  is locally compact Hausdorff.

Note that the underlying functor  $(\mathbf{Stone}) \rightarrow (\mathbf{Haus})$  factors through  $(\mathbf{CptHaus})$ . This implies that,  $X \rightarrow \underline{\text{alg}}(X)$  factors through  $\beta(X)$  for any Hausdorff space  $X$ . Usually,  $\underline{\text{alg}}(X)$  does not coincide with  $\beta(X)$ .

**Example 4.3.11.** (1) If  $X$  is a discrete space, then  $\beta(X)$  coincides with  $\underline{\text{alg}}(X)$ : indeed,  $C(X) = \prod_{x \in X} \mathbb{F}_1$ , and there is a bijection between the set of points of  $\text{Spec } C(X)_{\text{cpt}}$  and set of ultrafilters on  $X$ . This also coincides with the set of points of  $\beta(X)$ . We can also see that  $\beta(X) \rightarrow \underline{\text{alg}}(X)$  is in fact, a homeomorphism.

- (2) The real line  $\mathbb{R}$  is a locally compact Hausdorff space, hence it becomes an open subset of  $\beta\mathbb{R}$ . On the other hand,  $\underline{\text{alg}}(\mathbb{R})$  is a one point set, since  $\mathbb{R}$  is connected, but Stone spaces are totally disconnected.

## 5 Schemes

### 5.1 Sheaves

When considering sheaves of complete-algebra valued, we need some special care, for there are some obstructions when applying the sheaf theory to algebras admitting infinite operations (in our case, the supremum map.): we don't have sheafifications in general, and the stalk of such sheaves do not admit a natural induced infinite operators. Thus, the notion of algebraicity is essential in the following.

**Definition 5.1.1.** Let  $X$  be a topological space.

- (1) We regard the preordered set  $C(X)$  as a category.

- (2) Let  $\mathcal{A}$  be a category. A functor  $\mathcal{F} : C(X)^{\text{op}} \rightarrow \mathcal{A}$  is a  $\mathcal{A}$ -valued *presheaf* on  $X$ . A morphism of presheaves on  $X$  is a natural transformation. A presheaf is a *sheaf* if  $\mathcal{F}$  is a continuous functor.
- (3) We denote by  $(\mathcal{A}\text{-}\mathbf{PSh}/X)$  (resp.  $(\mathcal{A}\text{-}\mathbf{Sh}/X)$ ) the category of  $\mathcal{A}$ -valued presheaves (resp. sheaves) on  $X$ .

Here, we chose the definition of a sheaf to be a contravariant functor from the category  $C(X)$  of closed subsets of  $X$ : this is just for convenience sake.

**Definition 5.1.2.** Let  $\sigma$  be a pre-complete algebraic type and  $X$  be a topological space. Then, the underlying functor  $U : ((\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})\text{-}\mathbf{Sh}/X) \rightarrow ((\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})\text{-}\mathbf{PSh}/X)$  has a left adjoint  $S$ , which we call the *sheafification*.

*Proof.* This is obvious, since  $(\mathbf{alg}.\sigma^\dagger\text{-}\mathbf{alg})$  is equivalent to  $(\sigma\text{-}\mathbf{alg})$  by Proposition 1.1.7, and  $(\sigma\text{-}\mathbf{alg})$ -valued presheaves admit sheafifications.  $\square$

Next, we will see what happens when the underlying topological space is algebraic.

**Definition 5.1.3.** Let  $X$  be an algebraic sober space. Then,  $C(X)_{\text{cpt}}$  becomes a pre-complete idealic semiring with idempotent multiplication. Let  $\mathcal{A}$  be any category.

- (1) We call a functor  $C(X)_{\text{cpt}}^{\text{op}} \rightarrow \mathcal{A}$  a  $\mathcal{A}$ -valued *presheaf* on  $X^{\text{cpt}}$ . It is a *sheaf* if it is finite continuous, i.e. preserves fiber products.
- (2) We denote by  $(\mathcal{A}\text{-}\mathbf{PSh}/X^{\text{cpt}})$  (resp.  $(\mathcal{A}\text{-}\mathbf{Sh}/X^{\text{cpt}})$ ) the category of  $\mathcal{A}$ -valued presheaves (resp. sheaves) on  $X^{\text{cpt}}$ .

**Proposition 5.1.4.** Let  $X$  be an algebraic sober space, and  $\mathcal{A}$  be a small complete category. Then, the underlying functor  $U^{\text{cpt}} : (\mathcal{A}\text{-}\mathbf{Sh}/X) \rightarrow (\mathcal{A}\text{-}\mathbf{Sh}/X^{\text{cpt}})$  is an equivalence of categories.

*Proof.* Let  $\mathcal{F}$  be a  $\mathcal{A}$ -valued sheaf on  $X^{\text{cpt}}$ . We define a  $\mathcal{A}$ -valued sheaf  $\mathcal{F}^\dagger$  on  $X$  by

$$\mathcal{F}^\dagger(Z) = \varprojlim_{Z' \leq Z} \mathcal{F}(Z'),$$

where  $Z'$  runs through all the compact elements smaller than  $Z$ . We claim that  $\mathcal{F}^\dagger$  is indeed a sheaf.

For any  $Z \in C(X)$ , any covering  $Z = \sum Z_\lambda$  and  $a_\lambda \in \mathcal{F}^\dagger(Z_\lambda)$  satisfying  $a_\lambda|_{Z_\lambda Z_\mu} = a_\mu|_{Z_\lambda Z_\mu}$ , we will show that there exists a unique  $a \in \mathcal{F}^\dagger(Z)$  such that  $a|_{Z_\lambda} = a_\lambda$ . We have  $a_\lambda|_{WC_\lambda C_\mu} = a_\mu|_{WC_\lambda C_\mu}$  for any compact  $W \leq Z$  and  $C_\lambda \leq Z_\lambda$ . Since  $\mathcal{F}$  is a sheaf and  $W$  is covered by finitely many  $WC_\lambda$ 's, there is a unique  $a_W \in \mathcal{F}^\dagger(W)$  such that  $a_W|_{WC_\lambda} = a_\lambda|_{WC_\lambda}$ . By the definition of  $\mathcal{F}^\dagger$ , the  $a_W$ 's patch together to give a section  $a \in \mathcal{F}^\dagger(Z)$ . It is clear that  $a|_{Z_\lambda} = a_\lambda$ . Uniqueness of  $a$  is also clear from the construction.

Given a morphism  $f : \mathcal{F} \rightarrow \mathcal{G}$  of sheaves on  $X^{\text{cpt}}$ , we define  $f^\dagger : \mathcal{F}^\dagger \rightarrow \mathcal{G}^\dagger$  by

$$\mathcal{F}^\dagger(Z) = \varprojlim_{Z'} \mathcal{F}(Z') \xrightarrow{f} \varprojlim_{Z'} \mathcal{G}(Z') = \mathcal{G}^\dagger(Z).$$

This is well defined. Hence, we have a functor  $\underline{\text{comp}} : (\mathcal{A}\text{-}\mathbf{Sh}/X^{\text{cpt}}) \rightarrow (\mathcal{A}\text{-}\mathbf{Sh}/X)$ .

We will see that this is the left adjoint of  $\overline{U}^{\text{cpt}}$ . The unit  $\epsilon : \text{Id}_{(\mathcal{A}\text{-}\mathbf{Sh}/X^{\text{cpt}})} \Rightarrow U^{\text{cpt}} \underline{\text{comp}}$  is given by the natural isomorphism  $\mathcal{F}(Z) \simeq \mathcal{F}^\dagger(Z)$  for any compact  $Z$ . The counit  $\eta : \underline{\text{comp}} U^{\text{cpt}} \rightarrow \text{Id}_{(\mathcal{A}\text{-}\mathbf{Sh}/X)}$  is given by the natural isomorphism

$$(\mathcal{G}^{\text{cpt}})^\dagger(Z) = \varprojlim_{Z' \leq Z : \text{cpt}} \mathcal{G}(Z') \simeq \mathcal{G}(Z).$$

□

## 5.2 Idealic schemes

Here, we will introduce a notion of idealic schemes.

The significant difference between idealic schemes and the usual scheme is that we can construct a universal idealic scheme from its underlying space.

**Proposition 5.2.1.** Let  $X$  be a topological space,  $Z$  be a closed subset of  $X$ , and  $U = X \setminus Z$  be the complement of  $Z$ . Then, the restriction homomorphism  $\pi : C(X) \rightarrow C(U)$  induces an isomorphism  $C(X)_Z \simeq C(U)$ , where  $C(X)_Z$  is the localization along  $Z$ .

*Proof.* Since  $\pi(Z) = 1$  in  $C(U)$ ,  $\pi$  factors through  $C(X)_Z$ :

$$\begin{array}{ccc} C(X) & \xrightarrow{\pi} & C(U) \\ \downarrow & \nearrow \tilde{\pi} & \\ C(X)_Z & & \end{array}$$

It is clear that  $\pi$  is surjective, hence so is  $\tilde{\pi}$ . It remains to show that  $\tilde{\pi}$  is injective. Let  $\mathfrak{a}$  be the congruence relation generated by  $(1, Z)$ . Then we see that

$$\mathfrak{a} = \{(a, b) \in C(X) \times C(X) \mid aZ = bZ\}.$$

Indeed, the righthand side contains  $(1, Z)$ , and it is easy to see that this is a congruence relation, since the multiplication is idempotent. If  $\pi(a) = \pi(b)$ , then  $aZ = a \cup Z = b \cup Z = bZ$ , hence  $a$  coincides with  $b$  in  $C(X)_Z$ . □

**Definition 5.2.2.** A functor  $\tau_X : C(X)^{\text{op}} \rightarrow (\mathbf{IRng}^\dagger)$  is a sheaf on  $X$  defined by  $Z \mapsto C(X)_Z \simeq C(X \setminus Z)$ .

**Definition 5.2.3.** Let  $\alpha : (\mathbf{IRng}^\dagger) \rightarrow (\mathbf{IIRng}^\dagger)$  be the left adjoint of the underlying functor  $(\mathbf{IIRng}^\dagger) \rightarrow (\mathbf{IRng}^\dagger)$ .

(1) A triple  $X = (X, \mathcal{O}_X, \beta_X)$  is an *idealic scheme* if:

- (a)  $X$  is a sober space.
- (b)  $\mathcal{O}_X$  is  $(\mathbf{IRng}^\dagger)$ -valued sheaf.
- (c)  $\beta_X : \alpha \mathcal{O}_X \rightarrow \tau_X$  is a morphism of  $(\mathbf{IIRng}^\dagger)$ -valued presheaves.

- (d) The restriction maps reflect localization: for any  $W \leq Z$  of closed subsets of  $X$ , and any  $a \in \mathcal{O}_X(Z)$  satisfying  $\beta_X(\alpha(a)) \geq W$ , the restriction functor  $\mathcal{O}_X(Z) \rightarrow \mathcal{O}_X(W)$  factors through the localization  $\mathcal{O}_X(Z)_a$ .

$$\begin{array}{ccc} \mathcal{O}_X(Z) & \xrightarrow{\text{res}} & \mathcal{O}_X(W) \\ \downarrow & \nearrow & \\ \mathcal{O}_X(Z)_a & & \end{array}$$

- (2) A *morphism*  $(X, \mathcal{O}_X, \beta_X) \rightarrow (Y, \mathcal{O}_Y, \beta_Y)$  of *idealistic schemes* is a pair  $(f, f^\#)$  such that:

- (a)  $f : X \rightarrow Y$  is a continuous map.
- (b)  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  is a morphism of  $(\mathbf{IRng}^\dagger)$ -valued sheaves on  $Y$ .
- (c) The following diagram of  $(\mathbf{IIRng}^\dagger)$ -valued presheaves is commutative:

$$\begin{array}{ccc} \alpha \mathcal{O}_Y & \xrightarrow{\alpha f^\#} & \alpha f_* \mathcal{O}_X \\ \beta_Y \downarrow & & \downarrow f_* \beta_X \\ \tau_Y & \xrightarrow{C(f)} & f_* \tau_X \end{array}$$

- (3) We denote by  $(\mathbf{ISch})$  the category of idealistic schemes.

**Proposition 5.2.4.** The underlying functor  $U : (\mathbf{ISch}) \rightarrow (\mathbf{Sob})$  has a right adjoint.

*Proof.* We will construct a functor  $C^+ : (\mathbf{Sob}) \rightarrow (\mathbf{ISch})$ . Given a sober space  $X$ , set  $\mathcal{O}_X = \tau_X$ , and let  $\beta_X$  be the identity. Then  $C^+(X) = (X, \tau_X, \text{Id})$  becomes an idealistic scheme. Given a continuous map  $f : X \rightarrow Y$  of sober spaces, set  $f^\# = C(f) : \tau_Y \rightarrow f_* \tau_X$ . This gives a morphism  $(f, f^\#) : C^+(X) \rightarrow C^+(Y)$  of idealistic schemes. Hence, a functor  $C^+ : (\mathbf{Sob}) \rightarrow (\mathbf{ISch})$  is well defined.

We will show that this is the right adjoint of the underlying functor  $U$ . The unit  $\epsilon : \text{Id}_{(\mathbf{ISch})} \Rightarrow C^+ \circ U$  is defined by  $\epsilon(X) = (\text{Id}_X, \beta_X \alpha)$ . The counit  $U \circ C^+ \rightarrow \text{Id}_{(\mathbf{Sob})}$  is given by the identity.  $\square$

**Remark 5.2.5.** Our next goal is to construct a left adjoint of the global section functor  $\Gamma : (\mathbf{ISch})^{\text{op}} \rightarrow (\mathbf{IRng}^\dagger)$ . However, this seems to be impossible in general, due to what we mentioned in the beginning of the previous subsection.

### 5.3 $\mathcal{A}$ -schemes

We will introduce a notion of  $\mathcal{A}$ -schemes, generalizing the classical schemes in algebraic geometry. This is because several different type of schemes began to appear lately, and we thought there should be a general theory how to construct these schemes.

**Definition 5.3.1.** Let  $\sigma$  be an algebraic type with a multiplicative monoid structure.

- (1) A  $\sigma$ -algebra with a multiplicative system is a pair  $(R, S)$ , where  $R$  is a  $\sigma$ -algebra and  $S$  is a multiplicative submonoid of  $R$ .
- (2) A homomorphism  $(A, S) \rightarrow (B, T)$  of  $\sigma$ -algebras with multiplicative systems is a homomorphism  $f : A \rightarrow B$  of  $\sigma$ -algebras satisfying  $f(S) \subset T$ .
- (3) We denote by  $(\sigma + \mathbf{Mnd})$  the category of  $\sigma$ -algebras with multiplicative systems.
- (4) For an object  $(R, S) \in (\sigma + \mathbf{Mnd})$ , define  $L(R) = R_S$  as the localization of  $R$  along  $S$ . Note that localization exists in  $(\sigma\text{-}\mathbf{alg})$  for any algebraic type  $\sigma$ . Given a homomorphism  $f : (A, S) \rightarrow (B, T)$  of  $\sigma$ -algebras with multiplicative systems, we have a natural homomorphism  $L(f) : A_S \rightarrow B_T$ . Hence we obtain a functor  $L : (\sigma + \mathbf{Mnd}) \rightarrow (\sigma\text{-}\mathbf{alg})$ , to which we refer as the *localization functor*.

**Definition 5.3.2.** We will fix a quadruple  $\mathcal{A} = (\sigma, \alpha_1, \alpha_2, \gamma)$  in the following. Here,

- (1)  $\sigma$  is an algebraic type, with a multiplicative monoid structure: equivalently, there is an underlying functor  $U_1 : (\sigma\text{-}\mathbf{alg}) \rightarrow (\mathbf{Mnd})$  preserving multiplications.
- (2)  $\alpha_1 : (\sigma\text{-}\mathbf{alg}) \rightarrow (\mathbf{PIRng})$  is a functor, where  $(\mathbf{PIRng})$  is the category of pre-complete idealic semirings with idempotent multiplications.
- (3)  $\alpha_2 : U_1 \Rightarrow U_2 \alpha_1$  is a natural transformation, where  $U_2 : (\mathbf{PIRng}) \rightarrow (\mathbf{Mnd})$  is the underlying functor, preserving multiplications.
- (4) The pair  $\alpha = (\alpha_1, \alpha_2)$  gives a functor  $\alpha : (\sigma + \mathbf{Mnd}) \rightarrow (\mathbf{PIRng} + \mathbf{Mnd})$ , namely:  $\alpha(R, S) = (\alpha_1(R), \alpha_2(S))$ , and if  $f : (A, S) \rightarrow (B, T)$  is a homomorphism of  $\sigma$ -algebras with multiplicative systems, then  $(\alpha_1 f)(\alpha_2(S)) \subset \alpha_2(T)$ .
- (5)  $\gamma : L\alpha_1 \Rightarrow \alpha L$  is a natural isomorphism:

$$\begin{array}{ccc}
 (\sigma + \mathbf{Mnd}) & \xrightarrow{\alpha} & (\mathbf{PIRng} + \mathbf{Mnd}) \\
 L \downarrow & \xRightarrow{\gamma} & \downarrow L \\
 (\sigma\text{-}\mathbf{alg}) & \xrightarrow{\alpha_1} & (\mathbf{PIRng})
 \end{array}$$

We will refer to  $\mathcal{A}$  as a "schematizable algebraic type".

**Definition 5.3.3.** Let  $\mathcal{A}$  be as above.

- (1) For any  $\sigma$ -algebra  $R$  and any element  $Z \in \alpha_1(R)$ , we denote by  $R_Z = R_{\alpha_2^{-1}(Z)}$  the *localization of  $R$  along  $Z$* , where

$$\alpha_2^{-1}(Z) = \{x \in R \mid \alpha_2(x) \geq Z\}.$$

Note that  $\alpha_2^{-1}(Z)$  is a monoid.

- (2) A schematizable algebraic type  $\mathcal{A}$  satisfies the *strong patching condition* if the following holds:
- (i) For any  $\sigma$ -algebra  $R$ ,  $\alpha_2(R)$  generates  $\alpha_1(R)$  as a pre-complete idealic semiring.
  - (ii) Let  $R$  be any  $\sigma$ -algebra, and  $s, s_1, \dots, s_n$  be elements of  $R$  satisfying  $\alpha_2(s) = \sum \alpha_2(s_i)$  in  $\alpha_1(R)$ . If there are elements  $a_i \in R_{s_i}$  such that  $a_i = a_j$  in  $R_{s_i s_j}$ , then there exists a unique  $a \in R_s$  such that  $a = a_i$  in  $R_{s_i}$ .
- (3)  $\mathcal{A}$  satisfies the *weak patching condition*, if (i) holds, and (ii) holds for  $s = 1$ .

**Example 5.3.4.** Here are some examples of schematizable algebraic types: for any of them, the functor  $\alpha_1$  factors through **(PIRng)** (the category of pre-complete idealic semirings; but in this subsection, we will not assume the existence of infimum operators), so we will just describe  $\alpha'_1 : (\sigma\text{-alg}) \rightarrow (\mathbf{PIRng})$  for (1) and (2).

- (1) The algebraic type  $\sigma$  is that of rings, and  $\alpha'_1 : (\mathbf{Rng}) \rightarrow (\mathbf{PIRng})$  sends a ring  $R$  to the set of finitely generated ideals on  $R$ . Note that  $\alpha_1(R) = C(\text{Spec } R)_{\text{cpt}}$ . A homomorphism  $f : A \rightarrow B$  gives a homomorphism  $\alpha'_1(f) : \alpha'_1(A) \rightarrow \alpha'_1(B)$  defined by

$$\mathfrak{a} \mapsto f(\mathfrak{a})B.$$

The map  $\alpha_2 : R \rightarrow \alpha_1(R)$  sends an element  $a \in R$  to  $V(a)$ , namely the support of  $a$ . This gives a natural transformation  $\alpha_2 : U_1 \Rightarrow U_2 \alpha_1$ , and a functor  $\alpha = (\alpha_1, \alpha_2) : (\mathbf{Rng} + \mathbf{Mnd}) \rightarrow (\mathbf{PIRng} + \mathbf{Mnd})$ .

For any multiplicative system  $S$  of a ring  $R$ , there is a natural isomorphism  $\gamma : \text{Spec } R_S \simeq \alpha_1(R_S) \simeq \alpha_1(R)_{\alpha_2(S)}$ .

In this case, the strong patching condition holds.

This schematizable algebraic type is what we use in classical algebraic geometry.

- (2) The algebraic type  $\sigma$  is that of monoids, and  $\alpha'_1$  sends a monoid  $M$  to the set of its finitely generated ideals: an *ideal* of  $M$  is a non-empty set  $\mathfrak{a}$  of  $M$  satisfying  $ax \in \mathfrak{a}$  for any  $a \in \mathfrak{a}$ ,  $x \in M$ . The natural transformation  $\alpha_2 : M \rightarrow \alpha_1(M)$  is defined by the unit of the adjoint  $(\mathbf{Mnd}) \rightleftarrows (\mathbf{PIRng})$ , which sends  $a \in M$  to the saturated ideal  $\bar{a}$  generated by  $a$ .

There is a natural isomorphism  $\gamma : \alpha_1(M_S) \simeq \alpha_1(M)_{\alpha_2(S)}$ , since  $\alpha_1(M) = \alpha_1(M/M^\times)$  for any  $M$ .

The weak patching condition is automatically satisfied, since any monoid is *local*, i.e. it admits a unique maximal ideal consisting of all non-unit elements. See [Deit].

This schematizable algebraic type is what we use in schemes over  $\mathbb{F}_1$ . Note that, given a ring  $R$ , we can construct a scheme of this type by regarding  $R$  as a multiplicative monoid. Hence, we must fix the schematizable algebraic type to specify how we make schemes.

- (3) The algebraic type  $\sigma$  is that of pre-complete idealic semirings, and  $\alpha_1 : (\mathbf{PIRng}) \rightarrow (\mathbf{PIIRng})$  is the left adjoint of the underlying functor. Note that  $\alpha_1(R) = C(\mathrm{Spec} R)_{\mathrm{cpt}}$ .  $\alpha_2 : R \rightarrow \alpha_1(R)$  is the unit of the adjoint  $(\mathbf{PIRng}) \rightleftarrows (\mathbf{PIIRng})$ , defined by  $a \mapsto V(a)$ . There is a natural isomorphism  $\gamma : C(\mathrm{Spec} R_S) \simeq C(\mathrm{Spec} R)_{\alpha_2(S)}$ .

The strong patching condition is satisfied, which we already verified in Lemma 4.2.11. This schematizable algebraic type is used in tropical geometry.

**Proposition 5.3.5.** Let  $R$  be a pre-complete idealic semiring, and  $X = \mathrm{Spec} R$  be its spectrum. Define a  $(\mathbf{PIRng})$ -valued presheaf  $\mathcal{O}_X : C(X)_{\mathrm{cpt}}^{\mathrm{op}} \rightarrow (\mathbf{PIRng})$  on  $X^{\mathrm{cpt}}$  by  $Z \mapsto R_Z$ . Then  $\mathcal{O}_X$  is a sheaf.

This is a direct consequence of Lemma 4.2.11.

**Definition 5.3.6.** Let  $X$  be an algebraic sober space. Define a  $(\mathbf{PIIRng})$ -valued sheaf  $\tau'_X$  on  $C(X)_{\mathrm{cpt}}$  by

$$Z \mapsto C(X)_{\mathrm{cpt}, Z} \simeq C(X \setminus Z)_{\mathrm{cpt}}.$$

This is indeed a sheaf, by the above proposition.

We will give a definition of  $\mathcal{A}$ -schemes. Note that we are defining the structure sheaves on  $X^{\mathrm{cpt}}$ , where  $X$  is an algebraic sober space. This is sufficient, by Proposition 5.1.4.

**Definition 5.3.7.** Let  $\mathcal{A}$  be a schematizable algebraic type.

- (1) A triple  $X = (X, \mathcal{O}_X, \beta_X)$  is a  $\mathcal{A}$ -scheme if:
- (a)  $X$  is an algebraic sober space.
  - (b)  $\mathcal{O}_X$  is a  $(\sigma\text{-}\mathbf{alg})$ -valued sheaf on  $C(X)_{\mathrm{cpt}}^{\mathrm{op}}$ .
  - (c)  $\beta_X : \alpha_1 \mathcal{O}_X \rightarrow \tau'_X$  is a morphism of  $(\mathbf{PIIRng})$ -valued sheaves, where  $\alpha_1 \mathcal{O}_X$  is the sheafification of  $Z \mapsto \alpha_1(\mathcal{O}_X(Z))$ .
  - (d) The restriction map reflects localization: Let  $W \leq Z$  be any two elements of  $C(X)_{\mathrm{cpt}}$ , and  $a \in \mathcal{O}_X(Z)$  be a section satisfying  $\beta_X(\alpha_2(a)) \geq$

$W$ . Then, the restriction map  $\mathcal{O}_X(Z) \rightarrow \mathcal{O}_X(W)$  factors through  $\mathcal{O}_X(Z)_a$ :

$$\begin{array}{ccc} \mathcal{O}_X(Z) & \xrightarrow{\text{res}} & \mathcal{O}_X(W) \\ \downarrow & \nearrow & \\ \mathcal{O}_X(Z)_a & & \end{array}$$

(2) A *morphism*  $(X, \mathcal{O}_X, \beta_X) \rightarrow (Y, \mathcal{O}_Y, \beta_Y)$  of  $\mathcal{A}$ -schemes is a pair  $(f, f^\#)$  such that:

- (a)  $f : X \rightarrow Y$  is a continuous map.
- (b)  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  is a morphism of  $(\sigma\text{-alg})$ -valued sheaves on  $Y$ .
- (c) The following diagram of **(PIIRng)**-valued sheaves is commutative:

$$\begin{array}{ccc} \alpha_1 \mathcal{O}_Y & \xrightarrow{\alpha_1 f^\#} & \alpha_1 f_* \mathcal{O}_X \\ \beta_Y \downarrow & & \downarrow f_* \beta_X \\ \tau'_Y & \xrightarrow{C(f)_{\text{cpt}}} & f_* \tau'_X \end{array}$$

(3) We denote by **( $\mathcal{A}$ -Sch)** the category of  $\mathcal{A}$ -schemes.

**Definition 5.3.8.** We will construct a functor  $\text{Spec}^{\mathcal{A}} : (\sigma\text{-alg}) \rightarrow (\mathcal{A}\text{-Sch})^{\text{op}}$  as follows:

- (1) Given a  $\sigma$ -algebra  $R$ , set  $X = \text{Spec } \alpha_1(R)^\dagger$ . Note that  $C(X)_{\text{cpt}} = \alpha_1(R)$ . Define the structure sheaf  $\mathcal{O}_X : \alpha_1(R)^{\text{op}} \rightarrow (\sigma\text{-alg})$  as the sheafification of a presheaf  $\mathcal{O}'_X : Z \mapsto R_Z$ .

We will define a morphism  $\beta_X : \alpha_1 \mathcal{O}_X \rightarrow \tau'_X$  of **(PIIRng)**-valued sheaves: For any element  $Z$  of  $\alpha_1(R)$ , set

$$S = S(Z) = \{x \in R \mid \alpha_2(x) \geq Z\}.$$

Then, we have a map

$$\alpha_1 \mathcal{O}'_X(Z) = \alpha_1(R_S) \xrightarrow{\gamma} \alpha_1(R)_{\alpha_2(S)} \rightarrow \alpha_1(R)_Z = \tau'_X(Z).$$

Sheafifying the lefthand side, we obtain a morphism  $\beta_X : \alpha_1 \mathcal{O}_X \rightarrow \tau'_X$ .

It is obvious that the restriction map reflects localization. Hence, we obtain a  $\mathcal{A}$ -scheme  $\text{Spec}^{\mathcal{A}} R = (X, \mathcal{O}_X, \beta_X)$ .

- (2) Let  $\varphi : B \rightarrow A$  be a homomorphism of  $\sigma$ -algebras. Set  $X = \text{Spec}^{\mathcal{A}} A$  and  $Y = \text{Spec}^{\mathcal{A}} B$ . We will construct a morphism  $(f, f^\#) : X \rightarrow Y$  as follows. The quasi-compact continuous map  $f = \text{Spec}(\varphi) : X \rightarrow Y$ . Note that  $C(f)_{\text{cpt}} = \alpha_1(\varphi)$ . Let us define the morphism  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  of sheaves. For any  $Z \in C(Y)$ , the homomorphism  $\varphi : B \rightarrow A_{\alpha_1(\varphi)(Z)}$



sends any element of  $S(Z)$  to an invertible element, since  $\alpha_2$  is a natural transformation. Hence, it gives rise to

$$\mathcal{O}'_Y(Z) = B_Z \rightarrow A_{\alpha_1(\varphi)(Z)} = \mathcal{O}'_X(f^{-1}Z).$$

Sheafifying the both sides, we obtain  $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ . It is easy to see that the following diagram commutes:

$$\begin{array}{ccc} \alpha_1 \mathcal{O}_Y & \xrightarrow{\alpha_1 f^\#} & \alpha_1 f_* \mathcal{O}_X \\ \beta_Y \downarrow & & \downarrow f_* \beta_X \\ \tau'_Y & \xrightarrow{\alpha_1(\varphi)} & f_* \tau'_X \end{array}$$

Hence, we obtain a functor  $\text{Spec}^{\mathcal{A}} : (\sigma\text{-alg}) \rightarrow (\mathcal{A}\text{-Sch})$ , to which we refer to as the *spectrum functor*.

**Theorem 5.3.9.** Suppose  $\mathcal{A}$  satisfies the weak patching condition. Then, the spectrum functor  $\text{Spec}^{\mathcal{A}}$  is the left adjoint of the global section functor  $\Gamma : (\mathcal{A}\text{-Sch})^{\text{op}} \rightarrow (\sigma\text{-alg})$ .

*Proof.* First, we will define the unit  $\epsilon : \text{Id}_{(\sigma\text{-alg})} \Rightarrow \Gamma \text{Spec}^{\mathcal{A}}$ . Given a  $\sigma$ -algebra  $R$ , set  $X = \text{Spec}^{\mathcal{A}} R$ . Then define  $\epsilon$  by

$$R \simeq \Gamma(X, \mathcal{O}'_X) \rightarrow \Gamma(X, \mathcal{O}_X).$$

Next, we will define the counit  $\eta : \text{Spec}^{\mathcal{A}} \Gamma \rightarrow \text{Id}_{(\mathcal{A}\text{-Sch})^{\text{op}}}$ . Given a  $\mathcal{A}$ -scheme  $X = (X, \mathcal{O}_X, \beta_X)$ , set  $Y = \text{Spec}^{\mathcal{A}} \Gamma(X)$ . The continuous map  $\eta : X \rightarrow \text{Spec}^{\mathcal{A}} \Gamma(X)$  between the underlying spaces is induced by  $\Gamma(\beta_X) : \alpha_1 \Gamma(X) \rightarrow C(X)_{\text{cpt}}$ . The morphism  $\eta^\# : \mathcal{O}_Y \rightarrow \eta_* \mathcal{O}_X$  is defined as follows: for a given  $Z \in \alpha_1(\Gamma(X))$ , the restriction map  $\Gamma(X) \rightarrow \mathcal{O}_X(\eta^{-1}Z) = \mathcal{O}_X(\beta_X(Z))$  gives rise to a homomorphism  $\mathcal{O}'_Y(Z) = \Gamma(X)_Z \rightarrow \mathcal{O}_X(\eta^{-1}Z)$ , since the restriction map reflects localization. Sheafifying the lefthand side, we obtain the required  $\eta^\#$ . Thanks to the weak patching condition, all four natural transformations  $\epsilon\Gamma$ ,  $\text{Spec}^{\mathcal{A}} \epsilon$ ,  $\Gamma\eta$ , and  $\eta \text{Spec}^{\mathcal{A}}$  becomes isomorphisms.  $\square$

Here, we will focus on to the case when  $\mathcal{A}$  is induced from idealic semirings. This case has a special feature that, we can construct the structure sheaf from the underlying space.

**Definition 5.3.10.** Let  $\mathcal{A}$  be the schematizable algebraic type induced from pre-complete idealic semirings, introduced in Example 5.3.4(3).

- (1) We call  $\mathcal{A}$ -schemes "algebraic idealic schemes". We denote by  $(\mathbf{alg.ISch})$  the categories of algebraic idealic schemes.
- (2) We define a functor  $C^{++} : (\mathbf{alg.Sob}) \rightarrow (\mathbf{alg.ISch})$  as follows: For an algebraic sober space  $X$ , set  $C^{++}(X) = (X, \tau'_X, \text{Id})$ . For a morphism  $f : X \rightarrow Y$  of algebraic sober spaces, set  $C^{++}(f) = (f, C(f)^{\text{cpt}}) : C^{++}(X) \rightarrow C^{++}(Y)$ , where  $C(f)^{\text{cpt}} : \tau'_Y \rightarrow f_* \tau'_X$  is the morphism induced by  $f$ .

**Proposition 5.3.11.** The functor  $C^{++}$  is the right adjoint of the underlying functor  $U : (\mathbf{alg.ISch}) \rightarrow (\mathbf{alg.Sob})$ .

*Proof.* The unit  $\epsilon : \text{Id}_{(\mathbf{alg.ISch})} \Rightarrow C^{++}U$  is given by  $(\text{Id}_X, \beta_X \alpha_1)$  for any algebraic idealic scheme  $(X, \mathcal{O}_X, \beta_X)$ . The counit  $\eta : UC^{++} \Rightarrow \text{Id}_{(\mathbf{alg.Sob})}$  is given by the identity.  $\square$

**Remark 5.3.12.** (1) When the schematizable algebraic type  $\mathcal{A}$  is the one we introduced in Example 5.3.4(3), then the structure sheaf  $\mathcal{O}_X$  of  $X = \text{Spec}^{\mathcal{A}} R$  is the functor defined by  $Z \mapsto R_{[Z]}$  for  $Z \in C(X)$ . This follows from Lemma 4.2.11.

(2) Note that we don't have a natural underlying functor  $(\mathbf{alg.ISch}) \rightarrow (\mathbf{ISch})$ : The sheaves  $\tau$  and  $\tau'$  which represent the topological structure, are different.

Let us summarize all the categories and functors we have obtained so far: Let  $\mathcal{A} = (\sigma, \alpha_1, \alpha_2, \gamma)$  be a schematizable algebraic type. Then the functors are illustrated below: The pairs of arrows are adjoints. The equal signs are equivalences.

$$\begin{array}{ccccc}
 & & & & (\mathbf{alg.IRng}^\dagger) \xrightleftharpoons[\Gamma]{\text{Spec}} (\mathbf{alg.ISch})^{\text{op}} \\
 & & & \nearrow U' & \uparrow \text{C}^{++} \downarrow U \\
 (\mathbf{alg}.\sigma^\dagger\text{-alg}) & \xrightleftharpoons[\text{comp}]{U_{\text{cpt}}} (\sigma\text{-alg}) & \xrightleftharpoons[\Gamma]{\text{Spec}^{\mathcal{A}}} (\mathcal{A}\text{-Sch})^{\text{op}} & \xrightarrow{U} & (\mathbf{alg.IIRng}^\dagger) \xrightleftharpoons[C]{\text{Spec}} (\mathbf{alg.Sob})^{\text{op}} \\
 \uparrow \text{alg} \downarrow U & \nearrow U & & & \uparrow \text{alg} \downarrow U \\
 (\sigma^\dagger\text{-alg}) & & & & (\mathbf{IIRng}^\dagger) \xrightleftharpoons[C]{\text{Spec}} (\mathbf{Sob})^{\text{op}} \\
 & & & & \nearrow U \downarrow \text{sob} \\
 & & & & (\mathbf{ISch})^{\text{op}} \xrightleftharpoons[C^+]{C} (\mathbf{Top})^{\text{op}}
 \end{array}$$

As we see, the right half can be regarded as the categories of geometric objects, while the left half are those of algebraic objects. The most mysterious part is the functor  $U : (\mathcal{A}\text{-Sch}) \rightarrow (\mathbf{alg.IIRng}^\dagger)$  in the middle, and it has been (and will continue to be) the central subject of algebraic geometry, arithmetic geometry, and tropical geometry. If  $\mathcal{A}$  is one of the example in 5.3.4, then  $U$  factor through  $(\mathbf{alg.IRng}^\dagger)$ .

## 5.4 Comparison with classical schemes

Next, we will see the relation between the category of classical schemes and the category of new schemes which we have introduced. In the proceedings,  $\mathcal{A}$  is the schematizable algebraic type induced from rings, introduced in Example 5.3.4(1).

**Proposition 5.4.1.** Let  $(\mathbf{QSch})$  be the category of quasi-compact, quasi-separated schemes and quasi-compact morphisms, in the classical sense. Then, there is a fully faithful functor  $I : (\mathbf{QSch}) \rightarrow (\mathcal{A}\text{-}\mathbf{Sch})$  which satisfies  $I \operatorname{Spec}^{\operatorname{op}} \simeq \operatorname{Spec}^{\mathcal{A}, \operatorname{op}}$ .

$$\begin{array}{ccc} (\mathbf{Rng})^{\operatorname{op}} & \xrightarrow{\operatorname{Spec}^{\operatorname{op}}} & (\mathbf{QSch}) \\ & \searrow \operatorname{Spec}^{\mathcal{A}, \operatorname{op}} & \downarrow I \\ & & (\mathcal{A}\text{-}\mathbf{Sch}) \end{array}$$

*Proof.* Let  $X = (X, \mathcal{O}_X)$  be a quasi-compact, quasi-separated scheme in the classical sense. We must construct a morphism  $\beta_X : \alpha_1 \mathcal{O}_X \rightarrow \tau'_X$  of sheaves. Since  $X$  is locally isomorphic to an affine scheme, we already have a local isomorphism  $\alpha_1 \mathcal{O}_X \simeq \tau'_X$ , which patch up to give a global morphism  $\beta_X$ . Hence we obtain a  $\mathcal{A}$ -scheme  $(X, \mathcal{O}_X, \beta_X)$ .

For any quasi-compact morphism  $f : X \rightarrow Y$  of quasi-compact, quasi-separated schemes,  $f$  commutes with  $\beta$ , since  $f$  is locally induced by a homomorphism of rings. This means that  $f$  naturally becomes a morphism of  $\mathcal{A}$ -schemes. Hence, we have a functor  $I : (\mathbf{QSch}) \rightarrow (\mathcal{A}\text{-}\mathbf{Sch})$ .

We claim that  $I$  is fully faithful. Let  $X, Y$  be quasi-compact, quasi-separated schemes, and  $f : I(X) \rightarrow I(Y)$  be a morphism of  $\mathcal{A}$ -schemes. We already know that  $I(X)$  and  $I(Y)$  are locally ringed spaces, and it suffices to show that  $f$  is a morphism of locally ringed spaces, i.e.  $\mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$  is a local homomorphism for any  $x \in X$ . This is a local argument, hence we may assume  $X$  and  $Y$  are both affine, say  $X = \operatorname{Spec} A$  and  $Y = \operatorname{Spec} B$ . Using Theorem 5.3.9, we have

$$\operatorname{Hom}_{(\mathcal{A}\text{-}\mathbf{Sch})}(I(X), I(Y)) \simeq \operatorname{Hom}_{(\mathbf{Rng})}(B, A) \simeq \operatorname{Hom}_{(\mathbf{QSch})}(X, Y),$$

which shows that  $f$  is induced locally (and hence globally) from a morphism of quasi-compact, quasi-separated schemes.

It is clear from the construction of  $\operatorname{Spec}^{\mathcal{A}}$  that  $I \operatorname{Spec}^{\operatorname{op}} \simeq \operatorname{Spec}^{\mathcal{A}, \operatorname{op}}$ .  $\square$

**Remark 5.4.2.** Let  $(\mathcal{M}\text{-}\mathbf{Sch})$  be the category of quasi-compact, quasi-separated schemes over  $\mathbb{F}_1$ , in the sense of Deitmar [Deit]. and  $\mathcal{A}$  be the schematizable algebraic type of Example 5.3.4(2). Then, by the same arguments as above, we obtain a fully faithful functor  $(\mathcal{M}\text{-}\mathbf{Sch}) \rightarrow (\mathcal{A}\text{-}\mathbf{Sch})$ .

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